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CHARACTERIZATION OF COMPOSITION OPERATOR FROM l_q INTO A BANACH SPACE

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Abstract

In this paper, we study composition operators on a Banach space of analytic functions, which includes the Bloch space. We presented operators from l_q into a Banach space of type p. We characterize a composition operator from l_q into a Banach space.

Keywords: Compact Operators, Composition Operators, Banach Space.

1. Introduction

In mathematics, the composition operator with symbol C_ϕ is a linear operator defined by the rule $C_\phi(f) = f \circ \phi,$ for f in the Banach space. This operator is formally linear:

$$(af + bg) \circ \phi = af \circ \phi + bg \circ \phi$$

Moreover, composition operators often come up in studying other operators. For all *f* belonging to a selected class. It is immediate to see that such an operator preserves harmonic mappings. In physics, and especially the area of dynamical systems, the composition operator is usually referred to as the Koopman operator. The domain of a composition operator can be taken more narrowly, as some Banach space, often consisting of holomorphic functions: for example, some Hardy space or Bergman space. In mathematics, composition operators commonly occur in the study of shift operators, for example, in the Beurling–Lax theorem and the Wold decomposition. Shift operators can be studied as one-dimensional spin lattices. Composition operators appear in the theory of Aleksandrov–Clark measures. A diagonal operator in the broad sense of the word is an operator D of multiplication by a complex function λ in the direct integral of Hilbert spaces. In functional analysis, a branch of mathematics, a compact operator is a linear operator T: X→ Y, where X,Y are normed vector spaces, with the property that T maps bounded subsets of X to relatively compact subsets of Y (subsets with compact closure in Y) [8,12]. Such an operator is necessarily a bounded operator, and so continuous. Some authors require that X,Y are Banach, but the definition can be

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extended to more general spaces. Any bounded operator T that has finite rank is a compact operator; indeed, the class of compact operators is a natural generalization of the class of finite-rank operators in an infinite-dimensional setting [4,7,8,9,11,12,14]. This study is arranged as follows. In section 1, we present background and fundamental information of diagonal operators. In section 2 we present operators from l_q into a Banach space of type p. In section 3, we characterize a composition operator from l_q into a Banach space.

1- Preliminaries

Let S be an operator admitting a factorization

$$l_q \overset{S}{\rightarrow} E$$

and

$$l_q \underset{D}{\rightarrow} l_1 \longrightarrow E$$

where D is a diagonal operator and T an arbitrary operator with the image in a Banach space of type p. We shall characterize these operators by entropy numbers. We give summability results for the eigenvalues of certain types of compact operators that are then applied to study integral operators.

The entropy numbers possess the following properties [1, 2]

- Monotonicity:

  $\|S\| = e_1(S) \geq e_2(S) \cdots \geq 0, \;\; for\; S \in \mathcal{L}(E, F).$

- Additivity:

  $e_{n+m}(S+T) \leq e_n(s) + e_m(T), \;\; for\;\; S, T \in \mathcal{L}(E, F).$

- Multiplicativity:

  $e_{n+m-1}(ST) \leq e_n(s) e_m(T), \;\; for\;\; T \in \mathcal{L}(E, F), S \in \mathcal{L}(E, G).$

Put

$$\mathcal{L}_{p,q} \coloneqq \left\{S \in \mathcal{L}: (e_n(S)) \in \mathcal{L}_{p,q}\right\}$$

and

$$L_{p,q}(S) \coloneqq \epsilon_{p,q} \left\|\left(e_n(S)\right)\right\|_{p,q}, \qquad for\;\; S \in \mathcal{L}'_{p,q}.$$

Where $0 < p, q < \infty$, stands for the quasi-normed Lorentz sequence space (cf. [3,8,9,12,15,16,21]), which for $p = q$ is the classical space of $p -$ summable sequences denoted by $\left[\mathcal{L}_p; || \;\; ||_p\right]$ and $\epsilon_{p,q}$ is a norming constant. Then $\mathcal{L}_{p,q};\; L_{p,q}$ becomes an injective and surjective quasi-normed operator ideal. [2,6,8,9,10,12,13].

From the multiplicativity of the entropy numbers we get the useful product formula

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$$\mathcal{L}_{p1,q1} \circ \mathcal{L}_{p0,q0} \subseteq \mathcal{L}_{p,q} \ , \ for \ \frac{1}{p} = \frac{1}{p0} + \frac{1}{p1} \ , \ \frac{1}{q} = \frac{1}{q0} + \frac{1}{q1}.$$

A Banach space E is type $p, 1 \leq p \leq 2$, if there is a constant $c(p, E)$ such that

$$\int_0^1 \left\| \sum_{i=1}^{n} r_i(t) x_i \right\| dt \leq c(P, E) \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}.$$

For $x_1, \cdots x_n \in E$, where $r_1, \cdots r_n$ are the Rademacher functions on [0,1].

There is a constant $c(p, E)$ such that for all independent E -valued random variables $z_1, \cdots, z_n, \ n = 1,2, \cdots$ with finite $p - th$ moment the inequality

$$\left\| \sum_{i=1}^{n} (z_i - \mathbb{E}\, z_i) \right\| \leq c(p, E) \left(\sum_{i=1}^{n} \mathbb{E} \|z_i\|^p \right)^{\frac{1}{p}},$$

holds, where $\mathbb{E}$ is the mathematical expectation.

2- Operators from l_q into a Banach Space of Type p

Definition 1

Let X be a bounded linear space $n \in \mathbb{N}$ and B be a subset of X. Then the quantity

$$\mathcal{E}_n(B, X) \coloneqq inf\{\varepsilon > 0 \colon B \ can \ be \ covered \ by \ 2^{n-1} \ balls \ with \ radius \ \varepsilon \ in \ X \}$$
$$= inf_{M_n} sup_{x \in B_X} inf_{y \in M_n} \|x - y\|. \quad (1)$$

It is called entropy number of B. Where $M_n$runs over all the subset in Y with $|M_n| < 2^{n-1}$.

Definition 2

Let $(X, \|\cdot\|)_X$ and $(Y, \| \cdot \|), Y$ be two normed linear spaces

$T \colon X \longrightarrow Y$ be abounded linear operator, and $n \in \mathbb{N}$. Then the quantity

$$\mathcal{E}_n(T) \coloneqq \mathcal{E}_n(T \colon X \longrightarrow Y) = \mathcal{E}_n(T(B_X), Y) \quad (2)$$

called the entropy number of operator T [8,9,12,17,18,19,20], where B_X is the unit ball of X.

From Definition 1 and Definition 2 we deduce the following lemmas:

Lemma 1

Let $\mathbb{E}$ be of type $p, \frac{1}{p'} + \frac{1}{p} = 1,$ and $S \in \mathcal{L}(l_1^m, E)$. Then,

$$e_k(S \colon l_1^m \longrightarrow E) \leq c(p, E) \|S \colon L_1^m \longrightarrow E\| k^{-1/p'} log^{2/p'}(3 . \tfrac{m}{k})$$

for $k = 1, \cdots, m$ where l_1^m denotes the m –dimentional vector space equipped with the norm $\|\cdot\|_1$.

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Lemma 2

Let E be of type $p, \frac{l}{p'} + \frac{1}{p} = 1$, and $S \in \mathcal{L}(l_1^m, E)$. Then

$$sup_{1 \le k \le \infty} k^{\frac{1}{s}} e_k(S: l_1^m \longrightarrow E) \le c(p,E) \|S: l_1^m \longrightarrow E\| m^{\frac{1}{s}-\frac{1}{p'}} . \qquad (3)$$

For $s < p', m = 1,2, \cdots$ ·

Proof

By using **Lemma 1** we get

$$sup_{1 \le k \le \infty} k^{\frac{1}{s}} e_k(S) \le \; c(p,E)\|S\| sup_{1 \le k \le m} k^{\frac{1}{s}-\frac{1}{p'}} log2^{2/p'}\left(3.\frac{em}{k}\right)$$
$$\le c_0(s,p,E)\|S\| m^{\frac{1}{s}-\frac{1}{p'}}$$

For $s < p'$.

Now, we estimate $sup_{k \le m} k^{\frac{1}{s}} e_k(S) \le$. For this purpose, let I_m denote the identity operator on an m- dimensional space. Because

$$sup_{k \le m} k^{\frac{1}{s}} e_k(S) \le sup_{k \ge 1}(m+k)^{\frac{1}{s}} e(m+k)(S)$$
$$\le sup_{k \ge 1}(m+k)^{\frac{1}{s}} e(m)(S: l_1^m \to E) e_k(I_m: l_1^m \to l_1^m)$$
$$\le e_m(S) sup_{k \ge 1} 2^{\frac{1}{s}} \left(m^{\frac{1}{s}} + k^{\frac{1}{s}}\right) e_k(I_m)$$
$$\le 2^{\frac{1}{s}} m^{\frac{1}{s}} e_m(S) + 2^{\frac{1}{s}} e_m(S) sup_{k \ge 1} k^{\frac{1}{s}} e_k(I_m)$$

and

$$sup_{k \ge 1} k^{\frac{1}{s}} e_k(I_m) \le \left(\sum_{\mathbf{1}}^{\infty} \boldsymbol{e_k^s}\,(\boldsymbol{I_m})\right)^{\frac{1}{s}}$$
$$\le 4\left(\sum_1^{\infty}\left(2^{-(k-1)/2m}\right)^{s}\right)^{\frac{1}{s}}$$
$$\le 4\left(\sum_1^{\infty}\left(2^{s/2m}\right)^{k-1}\right)^{\frac{1}{s}}$$
$$\le 4\frac{1}{\left(1-2^{-s/2m}\right)^{1/s}} \le 4\frac{2^{1/2m}}{\left(2^{s/2m}-1\right)^{1/s}}$$
$$\le \frac{82^{1/s}}{(sln(2))^{1/s}} m^{1/s},$$

where $e_k(l_m) \le 4 \cdot 2^{-k-1/2m}$ and $2^{s/2m} \ge 1 + (s/2m)\ln(2)$, then, we have

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$$e_m(S) \le c_1(P,E)\|S\|m^{-1/p'}.$$

The estimate

$$sup_{k>m}k^{\frac{1}{s}}e_k(S) \le c_2(S,P,E)\|S\|m^{\frac{1}{s}-\frac{1}{p'}}.$$

The preceding inequalities yield the required assertion via

$$sup_{1\le k\le\infty}k^{\frac{1}{s}}e_k(S) \le sup_{1\le k\le m}k^{\frac{1}{s}}e_k(S) + sup_{m<k}k^{\frac{1}{s}}e_k(S)$$
$$\le c(S,P,E)\|S\|m^{\frac{1}{s}-\frac{1}{p'}} \qquad for\ \ s<p'.$$

Which completes the proof.

3- Characterization of Composition Operator

In this section we characterized the composition operators from l_q into a Banach

Proposition 1. Let $\frac{1}{s}=\frac{1}{r}+\frac{1}{q}-max\left(\frac{1}{q};\frac{1}{2}\right).If\ r_0<r$, then there exists an operator $S\in\mathcal{L}(l_q,l_q)$ with

$$\sum_1^{\infty}\left\|S_{x_k}\right\|_q^r<\infty,\ \ and\ (\lambda_n(S))\notin\mathcal{L}_{s,r_0}.$$

Proof. For $1\le q\le 2$, we put $S=D\sim(\sigma_i)\in l_r\backslash l_{r,r_0}$. In the case $2\le q\le\infty$, let $T_{z^0}=(1)$,

$$T_{n^{n+1}}=\begin{pmatrix}T_{2^n} & T_{2^n}\\ T_{2^n} & -T_{2^n}\end{pmatrix}, \qquad n=1,2,\cdots$$

be the Littlewood matrices. Then, $T_{2^n}^2=2^nI_{2^n}$, where I_{2^n} is the identity operator on the 2n-dimensional vector space, $|\lambda_i(T_{2^n})|= 2^{n/2}\ ,for\ i=1,2,\cdots,2^n$. Choose a monotonically decreasing sequence $(\sigma_n)\in l_r\backslash l_{r_0}$ and define

$$S=\sum\sigma_n(2^n)-\frac{1}{r-1}/q\,T_2^n \coloneqq l_q(l_q^{2n})\to l_q(l_q^{2n})$$

As a block wise sum of multiples of the $T_{2^n}S$. Thus $S: l_q\to l_q$ and

$$\sum_1^{\infty}\|Sx_k\|_q^r=\sum|\sigma_n|^r(2^n)^{-1-\frac{r}{q(2^n)rq2^n}}.$$

$$\sum|\sigma_n|^r<\infty.$$

However, we have the following eigenvalue estimation,

$$\|(\lambda_i(S))\|_{s,r_0}^{r_0}\ge\sum|\lambda_i(S)|^{r_0}i^{\frac{r_0}{s}-1}$$
$$\ge|\sigma_n|_0^r=\infty.$$

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This means $(\lambda_n(S)) \notin \mathcal{L}_{s,r_0}$. For $r_0 < r$. Which completes the proof.

Corollary 1. Let E be of type $1+\epsilon, \epsilon \geq 0$, and $S_j \in \mathcal{L}(l_1^m, E_q)$. Then,

$$sup_{1\leq k<\infty} \sum_j k^{\frac{1}{s}} e_k(S_j : l_1^m \to E_j) \leq \sum_j c(s, 1+\epsilon, E_j) \| S_j : l_1^m \to E_j \| m^{1/s - \frac{\epsilon}{1+\epsilon}},$$

$$for\ s < \frac{1+\epsilon}{\epsilon}, m = 1,2,\cdots \cdot$$

Theorem 1. Let E be of type $p, T \in \mathcal{L}_{\boldsymbol{l_1},\boldsymbol{E}}$ and

$$D \in \mathcal{L}_{(\mathbf{1+\epsilon}),\boldsymbol{l_1}}, (\sigma_i) \in, 0 \leq \epsilon < \infty, \ \frac{\mathbf{1}}{(\mathbf{1+\epsilon})} + \frac{\mathbf{1}}{(\mathbf{1+\epsilon})} > 1.$$

Then, the composition operator $S = TD$ it is hold that

$$S \in \mathcal{L}_{\boldsymbol{l}(\mathbf{1+\epsilon}),\boldsymbol{E}}, \frac{1}{s} = \frac{2}{(1+\epsilon)} + \frac{1}{p}$$

Proof. We may write the diagonal operator D in the form

$$l_{(1+\epsilon)} \overset{D}{\to} l_1$$

and

$$l_{(1+\epsilon)} \underset{D_0}{\to} l_1 \underset{D_1}{\to} l_1$$

where the generating sequences (σ_i^0) of D_0 belong to $l_{(\mathbf{1+\epsilon})_\mathbf{0},\infty} \frac{\mathbf{1}}{(\mathbf{1+\epsilon})_\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{1+\epsilon})} > 1$, and (σ_i^1) of D_1, then we have

$$D_0 \in \mathcal{L}_{(\boldsymbol{s})_\mathbf{0},\infty}\left(l_{(\mathbf{1+\epsilon})}, l_1 \right) \ for \frac{\mathbf{1}}{\boldsymbol{s_0}} = \frac{\mathbf{1}}{(\mathbf{1+\epsilon})_\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{1+\epsilon})} - 1$$

And

$$TD_1 \in \mathcal{L}_{(\boldsymbol{s})_\mathbf{0},\infty}\left(l_{(\mathbf{1+\epsilon})}, l_1 \right), for \frac{\mathbf{1}}{\boldsymbol{s_0}} = \frac{\mathbf{1}}{(\mathbf{1+\epsilon})_\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{1+\epsilon})} - 1.$$

Thus

$$S \in \mathcal{L}_{(\boldsymbol{s})_\mathbf{1},(\mathbf{1+\epsilon})} \circ \mathcal{L}_{(\boldsymbol{s})_\mathbf{0},\infty}\left(l_{(\mathbf{1+\epsilon})}, E \right) \subseteq \mathcal{L}_{\boldsymbol{s},(\mathbf{1+\epsilon})}\left(l_{(\mathbf{1+\epsilon})}, E \right),$$

For

$$\frac{1}{s} = \frac{1}{s_0} + \frac{1}{s_1} = \frac{2}{(1+\epsilon)} - \frac{1}{p}.$$

Which completes the proof.

Corollary 2. Let E be of type $S \in \mathcal{L}(E, E)$ an operator which admits the factorization

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$$E \overset{D}{\rightarrow} E,\ \ 1 \leq \epsilon < \infty$$

where X, Y are arbitrary operators and D is a diagonal operator generated by a sequence $(\sigma_i) \in,\ \mathcal{L}_{(\mathbf{1+\epsilon}),(\mathbf{1+\epsilon})}, 0 \leq \epsilon < \infty,\ \frac{\mathbf{2}}{(\mathbf{1+\epsilon})} > 1$.

Then

$$(\lambda_i(S))\mathcal{L}_{(\mathbf{1+\epsilon})}, for\ \ \frac{1}{s} = \frac{2}{(1+\epsilon)} - min\left(\frac{1}{p}; max\left(\frac{1}{(1+\epsilon)}, \frac{1}{2}\right)\right).$$

Theorem 2. Let $\frac{1}{\mathrm{s}} = \frac{1}{(r_0+\epsilon)} + \frac{1}{(1+\epsilon)} - max\left(\frac{1}{(1+\epsilon)}, \frac{1}{2}\right)$. If $\epsilon \geq 0$. Then, there exists an operator $S \in \mathcal{L}\left(l_{(1+\epsilon)}, l_{(1+\epsilon)}\right)$ with

$$\sum_{1}^{\infty} \left\|S_{x_k}\right\|_{(1+\epsilon)}^{(r_0+\epsilon)} < \infty,\ \ and\ \ \left(\lambda_n(S)\right) \notin \mathcal{L}_{(s,r_0)}.$$

Proof. For $0 < \epsilon < 1$ we put we put $S = D \sim (\sigma_i) \in l_{(r_0+\epsilon)} \backslash l_{(r_0+\epsilon),,r_0}$. In the case $0 \leq \epsilon \leq \infty$, let $T_{z^0} = (1)$,

$$T_{n^{n+1}} = \begin{pmatrix} T_{2^n}^m & T_{2^n}^m \\ T_{2^n}^m & -T_{2^n}^m \end{pmatrix}, \qquad n = 1,2, \cdots$$

be the Littlewood matrices. Then, $T_{2^n}^m = 2^n I_{2^n}$, where I_{2^n} is the identity operator on the 2n-dimensional vector space, $|\lambda_i(T_{2^n}^m)| = 2^{n/2}\ , for\ i = 1,2, \cdots, 2^n$. Choose a monotonically decreasing sequence $(\sigma_n) \in l_{(r_0+\epsilon)} \backslash l_{r_0}$ and define

$$S^m = \sum \sigma_n(2^n) - \frac{1}{r_0+\epsilon} - \frac{1}{1+\epsilon} T_{2^n}^m = l_{(1+\epsilon)}\left(l_{(1+\epsilon)}^{2n}\right) \rightarrow l_{(1+\epsilon)}\left(l_{(1+\epsilon)}^{2n}\right).$$

As a block wise sum of multiples of the $T_{2^n}^m S$. Thus S^m: $l_{(1+\epsilon)} \rightarrow l_{(1+\epsilon)}$ and

$$\sum_n \|S^m x_k\|_{(1+\epsilon)}^{(r_0+\epsilon)} = \sum_n |\sigma_n|^{(r_0+\epsilon)} (2^n)^{-1-\frac{(r_0+\epsilon)}{(1+\epsilon)}(2^n)\frac{(r_0+\epsilon)}{(1+\epsilon)}(2^n)}.$$

$$\sum_n |\sigma_n|^{(r_0+\epsilon)} < \infty.$$

However, we have the following eigenvalue estimation,

$$\|(\lambda_i(S))\|_{s,r_0}^{r_0} \geq \sum |\lambda_i(S^m)|^{r_0} i^{\frac{\frac{r_0}{s}}{s}-1}$$

$$\geq |\sigma_n|^{r_0} = \infty.$$

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This means $(\lambda_n(S^m)) \notin \mathcal{L}_{s,r_0}$, for Let $\frac{1}{s} = \frac{1}{(r_0+\epsilon)} + \frac{1}{(1+\epsilon)} - \frac{1}{2}$ $and\ \epsilon \geq 0$. Which completes the proof.

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