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A NATURAL DUALITY FOR QUASI-VARIETY GENERATED BY A FIVE-ELEMENT RESIDUATED LATTICE

SYED MD OMAR FARUK
Associate Professor, Department of Mathematics, Shahjalal University of Science and Technology, Sylhet, Bangladesh. Email: omarfaruk-mat@sust.edu

Abstract

This research article explores the development of a natural duality for the quasi-variety generated by a specific five-element residuated lattice. Residuated lattices are algebraic structures that serve as the foundation for substructural logics, including relevance and linear logic. The establishment of natural dualities allows for a deeper understanding of these quasi-varieties by enabling representation via dual structures. In this paper we obtain the natural duality of the quasi-variety generated by a five-element residuated lattice.

Keywords: Natural Duality, Piggyback Duality, Residuated Lattice.

1. INTRODUCTION

Natural duality is fundamentally a category-theoretic concept. It is concerned with the quasi-variety generated by a finite algebra. The quasi-variety is of the form $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, the isomorphic copies of subalgebras of direct product of $\mathbf{M}$, for some finite algebra $\mathbf{M}$. The aim of the theory is to find a class $\mathcal{X} \coloneqq \mathbb{IS}_c\mathbb{P}^+(\widetilde{\mathbf{M}})$, the isomorphic copies of the closed substructures of non-zero powers of a topological structure $\widetilde{\mathbf{M}}$, having the same underlying set as $\mathbf{M}$, dual to $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$, which is in some sense simpler than $\mathcal{A}$. In this paper we are interested in obtaining a class dual to a quasi-variety generated by a five-element distributive residuated lattice. A complete account of natural duality theory is found in the monograp [1] by Clark and Davey.

The piggyback duality theorem is a very economical way to obtain a dual structure for distributive-lattice-based algebras. The application of the piggyback duality theorem to a distributive-lattice-based algebra is exhibited in this paper. We also refer the reader to applications of the piggyback technique for a Boolean algebra with a unary operation in [7] and for a three-element residuated chain in [9]. In Section 2 we include background on natural duality theory which we need in this paper. In Section 3 we apply the multi-carrier piggyback duality theorem to obtain a dual structure for the quasi-variety generated by a five-element distributive residuated lattice.

2. BACKGROUND OF NATURAL DUALITY

2.1 Definition of Natural Duality. Let $\mathbf{M} = < M; F >$ be a finite algebra and let $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$. Then $\mathcal{A}$ is the quasi-variety generated by $\mathbf{M}$. An **alter ego of $\mathbf{M}$** is a topological structure

$$\widetilde{\mathbf{M}} = < M; G, H, R, \mathcal{T} >$$

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On the same underline set as $\mathbf{M}$, such that:

(i) G is a set of **algebraic operations on $\mathbf{M}$**, that is, each $g \in G$ is a homomorphism $g: \mathbf{M}^n \to \mathbf{M}$ for some $n \in \mathbb{N}_0$;

(ii) H is a set of **algebraic partial operations on $\mathbf{M}$**, that is, each $h \in H$ is a homomorphism $g: \mathbf{D} \to \mathbf{M}$ where $\mathbf{D} < \mathbf{M}^n$ for some $n \in \mathbb{N}$;

(iii) R is a set of **algebraic relations on $\mathbf{M}$**, that is, each $r \in R$ is the underlying set of a subalgebra of $\mathbf{M}^n$ for some $n \in \mathbb{N}$;

(iv) $\mathcal{T}$ is the discrete topology on M.

Let $\widetilde{\mathbf{M}}$ be an alter ego of $\mathbf{M}$ and let $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\widetilde{\mathbf{M}})$. Then $\mathcal{X}$ is the topological quasi-variety generated by $\widetilde{\mathbf{M}}$. We refer the reader to [2] for details about topological quasi-varieties. As a categorical point of view, there is a natural way to obtain a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$ for each algebraic choice G, H and R. For every$\mathbf{A} \in \mathcal{A}$, define $D: \mathcal{A} \to \mathcal{X}$ such that $D(\mathbf{A})$ is the homset $\mathcal{A}(\mathbf{A}, \mathbf{M})$ viewed as a closed substructure of $\widetilde{\mathbf{M}}^A$. The structure $D(\mathbf{A})$ is called the **dual of $\mathbf{A}$,** for each $\mathbf{A} \in \mathcal{A}$. For every $\mathbf{X} \in \mathcal{X}$, define $E: \mathcal{X} \to \mathcal{A}$ such that $E(\mathbf{X})$ is the homset $\mathcal{X}(\mathbf{X}, \widetilde{\mathbf{M}})$ viewed as a subalgebra of $\mathbf{M}^X$. We now define D and E on morphisms. For $\varphi: \mathbf{A} \to \mathbf{B}$ where $\mathbf{A}, \mathbf{B} \in \mathcal{A}$, define $D(\varphi): D(\mathbf{B}) \to D(\mathbf{A})$ by $D(\varphi)(x) := x \circ \varphi$, and for $\psi: \mathbf{X} \to \mathbf{Y}$ where $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$, define by $E(\varphi): E(\mathbf{Y}) \to E(\mathbf{X})$ by $E(\varphi)(\alpha) := \alpha \circ \psi$. Then D and E are contravariant functors.

For each $\mathbf{A} \in \mathcal{A}$, there is a natural embedding $e_{\mathbf{A}}: \mathbf{A} \to ED(\mathbf{A})$, given by $e_{\mathbf{A}}(a)(x) = x(a)$, for all $a \in A$ and $x \in D(\mathbf{A})$. If $e_{\mathbf{A}}$ is an isomorphism for all $\mathbf{A} \in \mathcal{A}$, then we say that **$\widetilde{\mathbf{M}}$ yields a duality on $\mathcal{A}$.**

2.2 Piggyback Duality Theory. Piggyback duality theory is the first major step to obtain the dualising structure of a distributive-lattice-based algebra economically. The key ideas of the piggyback method were developed by Davey and Werner in [5, 6]. For the multi-sorted case we refer the reader to [3, 8]. Here we state the modification version which we need in this paper.

Let $\mathbf{2} :=< \{0, 1\}; \wedge, \vee, 0, 1 >$ be the two-element bounded distributive lattice and let $\mathcal{D} := \mathbb{ISP}(\mathbf{2})$ be the variety generated by $\mathbf{2}$. We shall say that $\mathbf{M}$ has a **term-reduct in $\mathcal{D}$** if there exist binary terms $\wedge$ and $\vee$ and constant unary terms u and z such that the algebra $\mathbf{M}^b := < M; \wedge^M, \vee^M, 0^M, 1^M >$ is a bounded distributive lattice, where 0^M and $1^M$are the values in M of the constant unary term functions z^M and u^M respectively.

Theorem 2.1 (**P**iggyback Duality Theory). Assume that $\mathbf{M}$ is a finite algebra which has a term-reduct in $\mathcal{D}$ and let $\mathcal{A} := \mathbb{ISP}\ (\mathbf{M})$. Let Ω be a set of $\mathcal{D}$-homomorphisms $\omega: \mathbf{M} \to \mathbf{2}$ and G be a set of $\mathcal{A}$-endomorphisms such that the set

$$\{\omega \circ g: \mathbf{M} \to \mathbf{2} \mid g \in G \text{ and } \omega \in \Omega\}$$

separates the points of M. Define R to be the set of all $\mathcal{A}$-subalgebras of $\mathbf{M}^2$ which are maximal in

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$$(\omega_1,\omega_2)^{-1}(\leq) := \{(a,b)\in M^2 | \omega_1(a)\leq \omega_2(b)\},$$

for some $\omega_1,\omega_2\in\Omega$. Then $\widetilde{\mathbf{M}} := < M; G\cup R, \mathcal{T} >$ yields a duality on $\mathcal{A}$.

The mappings $\omega\in\Omega$ are called **carriers** and the relations in the set R are referred to as the **piggyback relations**. We shall apply this theorem to obtain the dual structure of five-element residuated lattice.

In [5, 6], Davey and Werner developed the piggyback duality theory when a single $\mathcal{D}$-homomorphism ω is enough to separate the points of M. But this is not always possible. Unfortunately, in our case here we need more than one carrier to separate the points of M. In [3] Davey and Priestley introduce a set of $\mathcal{D}$-homomorphisms, so that the points of M can always be separated. Using Theorem 2.1, first we want to obtain a dualising structure for the quasi-variety generated by a five-element residuated lattice.

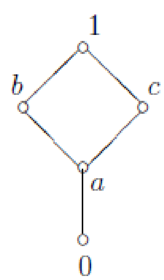
2.3 Residuated Lattice. An algebra $\mathbf{A} =< A; \vee, \wedge, \rightarrow, \cdot, 0, 1 >$ of type $< 2,2,2,2,0,0 >$ is a residuated lattice if

(i) $< A, \vee, \wedge, 0, 1 >$ is a bounded lattice with the greatest element 1 and least element 0;

(ii) $< A, \cdot, 1 >$ is a monoid;

(iii) $x\cdot y\leq z$ if and only if $x\leq y\rightarrow z$ for all $x,y,z\in A$.

Example 1. Let $\mathbf{M}_5 =< \{0,a,b,c,1\}; \vee, \wedge, \cdot, \rightarrow, 0, 1 >$ be the five element residuated lattice. The figure 1 represents the lattice corresponding to $\mathbf{M}_5$ where the operations $\cdot$ and $\rightarrow$ shown by the adjacent tables.



$\cdot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Figure 1: The Residuated lattice $\mathbf{M}_5$

3. PIGGYBACK DUALITY THEORY FOR $\mathbf{M}_5$

To apply the Piggyback Duality Theorem 2.1 on $\mathbf{M}_5$ we first have to determine G, the set of $\mathcal{A}$-endomorphisms, which separates the points of $\mathbf{M}_5$.

Lemma 3.1. The residuated lattice $\mathbf{M}_5$ has only the endomorphisms of $\mathrm{id}_{\mathbf{M}_5}$ and g where

$$g: 0\longmapsto 0, a\longmapsto 0, b\longmapsto 0, c\longmapsto 1, 1\longmapsto 1.$$

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Proof: To be a bounded lattice homomorphism we have either $\mathrm{id}_{\mathbf{M}_5}$ or

$g: 0 \longmapsto 0, a \longmapsto 0, b \longmapsto 0, c \longmapsto 1, 1 \longmapsto 1.$

$f: 0 \longmapsto 0, a \longmapsto b, b \longmapsto 1, c \longmapsto b, 1 \longmapsto 1.$

$h: 0 \longmapsto 0, a \longmapsto a, b \longmapsto 1, c \longmapsto a, 1 \longmapsto 1.$

$p: 0 \longmapsto 0, a \longmapsto c, b \longmapsto 1, c \longmapsto c, 1 \longmapsto 1.$

$q: 0 \longmapsto 0, a \longmapsto 0, b \longmapsto 1, c \longmapsto 0, 1 \longmapsto 1.$

$s: 0 \longmapsto 0, a \longmapsto 1, b \longmapsto 1, c \longmapsto 1, 1 \longmapsto 1.$

We now determine the homomorphisms which preserve the operations $\cdot$ and $\rightarrow$. The identity map always an endomorphism. Thus $\mathrm{id}_{\mathbf{M}_5}$ is an $\mathcal{A}$-endomorphism. Also the homomorphism g preserves the operations $\cdot$ and $\rightarrow$. Hence g is an $\mathcal{A}$-endomorphism. We have $f(a \rightarrow 0) = f(c) = b$ and $f(a) \rightarrow f(0) = b \rightarrow 0 = c$. Hence the lattice homomorphism f does not preserve $\rightarrow$. Therefore f is not an $\mathcal{A}$-endomorphism. We also have $h(a \cdot c) = h(a) = a$ and $h(a) \cdot h(c) = a \cdot a = 0$. Hence the lattice homomorphism h does not preserve $\cdot$. Therefore h is not an $\mathcal{A}$-endomorphism. Again we have $p(a \rightarrow 0) = p(c) = c$ and $p(a) \rightarrow p(0) = c \rightarrow 0 = 0$. Hence the lattice homomorphism p does not preserve $\rightarrow$. Therefore p is not an $\mathcal{A}$-endomorphism. Similarly we have $q(a \rightarrow 0) = q(c) = 0$ and $q(a) \rightarrow q(0) = 0 \rightarrow 0 = 1$. Hence the lattice homomorphism q does not preserve $\rightarrow$. Therefore q is not an $\mathcal{A}$-endomorphism. We have $s(a \cdot a) = s(0) = 0$ and $s(a) \cdot s(a) = 1 \cdot 1 = 1$. Hence the lattice homomorphism s does not preserve $\cdot$. Therefore s is not an $\mathcal{A}$-endomorphism.

Hence $\mathrm{id}_{\mathbf{M}_5}$ and are only the $\mathcal{A}$-endomorphisms.

The above Lemma shows that we need more than one carrier to separate the points of $\mathbf{M}_5$. Let $\omega_1: M_5 \rightarrow \mathbf{2}$ and $\omega_2: M_5 \rightarrow \mathbf{2}$ be two $\mathcal{D}$-homomorphism defined by

$$\omega_1 = \begin{cases} 1 & if\ x \in \{b, 1\} \\ 0 & if\ x \in \{0, a, c\} \end{cases}$$

And

$$\omega_2 = \begin{cases} 1 & if\ x \in \{a, b, c, 1\} \\ 0 & if\ x = 0 \end{cases}$$

It is easily seen that the ω_1 and ω_2 can separate the point of M_5. Hence $\Omega = \{\omega_1, \omega_2\}$. We now determine the subalgebra of $\mathbf{M}_5^2$ which are maximal in $(\omega_1, \omega_2)^{-1}(\leq)$ for each $\omega_1, \omega_2 \in \Omega$. The Figure 2, is the lattice structure of $\mathbf{M}_5^2$. The operations $\cdot$ and $\rightarrow$ are point wise on M_5^2. The maximal subalgebra in $(\omega_1, \omega_2)^{-1}(\leq)$ will be denoted $(\omega_1, \omega_2)^{-1}(\leq)^{\circ}$ by and we write $(\omega)^{-1}$ instead of $(\omega, \omega)^{-1}$. For convenience we shall write xy for $(x, y) \in M_5^2$.

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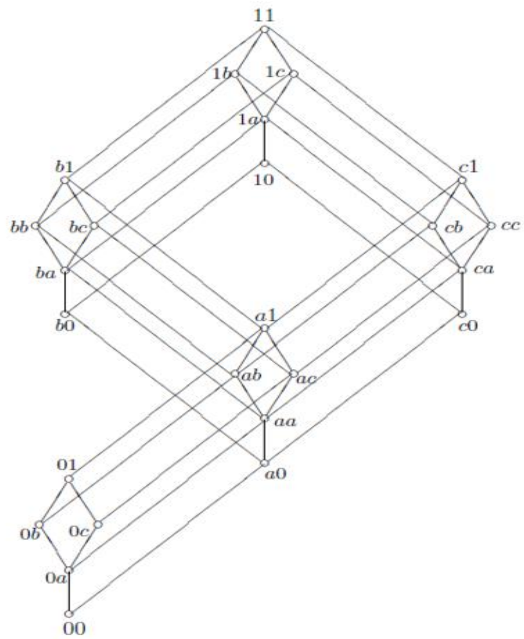

Figure 2: The algebra $\mathbf{M}_5^2$

Lemma 3.2 . Let $\mathbf{M}_5$ be the five element residuated lattice. Then

(i) $(\omega_1)^{-1}(\leq)^\circ =< \{00, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$.

(ii) $(\omega_1, \omega_2)^{-1}(\leq)^\circ =< \{00, 0a, 0b, 1c, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$.

(iii) $(\omega_2, \omega_1)^{-1}(\leq)^\circ =< \{00, c1, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$.

(iv) $(\omega_2)^{-1}(\leq)^\circ =< \{00, 0a, 0b, ab, bb, c1, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$.

Proof. (i) $(\omega_1)^{-1}(\leq) = \{xy \in M_5^2 \mid \omega_1(x) \leq \omega_1(y)\}$

$= M_5^2 \setminus \{b0, ba, bc, 10, 1a, 1c\}$

$= \{00, 0a, 0b, 0c, 01, a0, aa, ab, ac, a1, bb, b1, c0, ca, cb, cc, c1, 1b, 11\}$.

We want to show that

$$(\omega_1)^{-1}(\leq)^\circ =< \{00, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$$

is the maximal subalgebra. It is clear that $(\omega_1)^{-1}(\leq)^\circ$ is closed under $\vee$ and $\wedge$ (see Figure 2). Moreover from the Table 1, it is clear that $(\omega_1)^{-1}(\leq)^\circ$ is closed under $\cdot$ and $\rightarrow$. Thus $(\omega_1)^{-1}(\leq)^\circ$ is a subalgebra contained in $(\omega_1)^{-1}$. If we include $0a$, then we get $0a \rightarrow 00 = 1c$ which is not in $(\omega_1)^{-1}$. Again if we include $0b$, then $0b \rightarrow 00 = 1c$ which is not in $(\omega_1)^{-1}$.Similarly, it is easy to show that if we include any one of $0c,\ 01,\ a0,\ aa,\ ab, ac, a1$, $bb,\ b1,\ c0,\ ca,\ cb,\ cc,\ c1,\ 1b$, then to be a subalgebra $0c, 01, ac, b0, bc, c0, cc, c1$,10,$1c \in (\omega_1)^{-1}$ which is a contradiction.

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Hence $(\omega_1)^{-1}(\leq)^{\circ}$ is the maximal subalgebra contained in $(\omega_1)^{-1}$.

$\cdot$	00	11
00	00	00
11	00	11

The $\cdot$ table

$\to$	00	11
00	11	11
11	00	11

The $\to$ table

Table 1: Table for $(\boldsymbol{\omega_1})^{-1}(\leq)^{\circ}$

(ii) $(\omega_1,\omega_2)^{-1}(\leq) = \{xy \in M_5^2 \mid \omega_1(x) \leq \omega_2(y)\}$

$= M_5^2 \setminus \{10, b0\}$

$= \{00, 0a, 0b, 0c, 01, a0, aa, ab, ac, a1, ba, bb, bc, b1, c0, ca, cb,$
$cc, c1,1a, 1b, 1c, 11\}.$

We want to show that

$(\omega_1,\omega_2)^{-1}(\leq)^{\circ} =<\{00, 0a, 0b, 1c, 11\}; \vee,\wedge,\cdot,\to, 00, 11>$

is the maximal subalgebra. It is clear that $(\omega_1,\omega_2)^{-1}(\leq)^{\circ}$ is closed under $\vee$ and $\wedge$ (see Figure 2). Moreover from the Table 2, it is clear that $(\omega_1,\omega_2)^{-1}(\leq)^{\circ}$ is closed under $\cdot$ and $\to$. Thus $(\omega_1,\omega_2)^{-1}(\leq)^{\circ}$ is a subalgebra contained in $(\omega_1,\omega_2)^{-1}$. If we include $0c$, then we get $0c \to 00 = 10$ which is not in $(\omega_1,\omega_2)^{-1}$. Again if we include 01, then $01 \to 00 = 10$ which is not in $(\omega_1,\omega_2)^{-1}$. Similarly, it is easy to show that if we include any one of $a0, aa, ab, ac, a1, ba, bb, bc, b1, c0, ca, cb, cc, c1,1a, 1b$, then to be a subalgebra $0c, 01, ac, a1, b0, ba, c0, ca, cb, cc, c1,10,1a \in (\omega_1,\omega_2)^{-1}$ which is a contradiction.

Hence $(\omega_1,\omega_2)^{-1}(\leq)^{\circ}$ is the maximal subalgebra contained in $(\omega_1,\omega_2)^{-1}$.

$\cdot$	00	$0a$	$0b$	$1c$	11
00	00	00	00	00	00
$0a$	00	00	$0a$	$0a$	$0a$
$0b$	00	$0a$	$0b$	$0a$	$0b$
$1c$	00	$0a$	$0a$	$1c$	$1c$
11	00	$0a$	$0b$	$1c$	11

The $\cdot$ table

$\to$	00	$0a$	$0b$	$1c$	11
00	11	11	11	11	11
$0a$	$1c$	11	11	11	11
$0b$	$1c$	$1c$	11	$1c$	11
$1c$	00	$0b$	$0b$	11	11
11	00	$0a$	$0b$	$1c$	11

The $\to$ table

Table 2: Table for $(\boldsymbol{\omega_1},\boldsymbol{\omega_2})^{-1}(\leq)^{\circ}$

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(iii) $(\omega_2,\omega_1)^{-1}(\leq) = \{xy \in M_5^2 \mid \omega_2(x) \leq \omega_1(y)\}$

$= M_5^2 \setminus \{a0, aa, ac, b0, ba, bc, c0, ca, cc, 10, 1a, 1c\}$

$= \{00, 0a, 0b, 0c, 01, ab, a1, bb, b1, cb, c1, 1b, 11\}.$

We want to show that

$$(\omega_2,\omega_1)^{-1}(\leq)^{\circ} =< \{00, c1, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$$

is the maximal subalgebra. It is clear that $(\omega_2,\omega_1)^{-1}(\leq)^{\circ}$ is closed under $\vee$ and $\wedge$ (see Figure 2). Moreover from the Table 3, it is clear that $(\omega_2,\omega_1)^{-1}(\leq)^{\circ}$ is closed under $\cdot$ and $\rightarrow$. Thus $(\omega_2,\omega_1)^{-1}(\leq)^{\circ}$ is a subalgebra contained in $(\omega_2,\omega_1)^{-1}$. If we include $0a$, then we get $0a \rightarrow 00 = 1c$ which is not in $(\omega_2,\omega_1)^{-1}$. Again if we include $0b$, then $0b \rightarrow 00 = 1c$ which is not in $(\omega_2,\omega_1)^{-1}$. Similarly, it is easy to show that if we include any one of $0c, 01, ab, a1, bb, b1, cb, 1b$, then to be a subalgebra $0c, c0, cc, 10 \in (\omega_2,\omega_1)^{-1}$ which is a contradiction.

Hence $(\omega_2,\omega_1)^{-1}(\leq)^{\circ}$ is the maximal subalgebra contained in $(\omega_2,\omega_1)^{-1}$.

$\cdot$	00	$c1$	11
00	00	00	00
$c1$	00	$c1$	$c1$
11	00	$c1$	11

The $\cdot$ table

$\rightarrow$	00	$c1$	11
00	11	11	11
$c1$	00	11	11
11	00	$c1$	11

The $\rightarrow$ table

Table 3: Table for $(\omega_2,\omega_1)^{-1}(\leq)^{\circ}$

(iv) $(\omega_2)^{-1}(\leq) = \{xy \in M_5^2 \mid \omega_2(x) \leq \omega_2(y)\}$

$= M_5^2 \setminus \{a0, b0, c0, 10\}$

$= \{00, 0a, 0b, 0c, 01, aa, ab, ac, a1, ba, bb, bc, b1, ca, cb,$

$cc, c1, 1a, 1b, 1c, 11\}.$

We want to show that

$$(\omega_2)^{-1}(\leq)^{\circ} =< \{00, 0a, 0b, ab, bb, c1, 11\}; \vee, \wedge, \cdot, \rightarrow, 00, 11 >$$

is the maximal subalgebra. It is clear that $(\omega_2)^{-1}(\leq)^{\circ}$ is closed under $\vee$ and $\wedge$ (see Figure 2). Moreover from the Table 4, it is clear that $(\omega_2)^{-1}(\leq)^{\circ}$ is closed under $\cdot$ and $\rightarrow$. Thus $(\omega_2)^{-1}(\leq)^{\circ}$ is a subalgebra contained in $(\omega_2)^{-1}$. If we include $0c$, then we get $0c \rightarrow 00 = 10$ which is not in $(\omega_2)^{-1}$. Again if we include 01, then $01 \rightarrow 00 = 10$ which is not in $(\omega_2)^{-1}$.Similarly, it is easy to show that if we include any one of $aa, ac, a1, ba, bc, b1, ca, cb, cc, 1a, 1b, 1c$, then to be a subalgebra $0c, 01, a0, aa, a1, ba, b1, c0, ca, 1a \in (\omega_2)^{-1}$ which is a contradiction.

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Hence $(\omega_2)^{-1}(\leq)^{\circ}$ is the maximal subalgebra contained in $(\omega_2)^{-1}$.

$\cdot$	00	0a	0b	ab	bb	c1	11
00	00	00	00	00	00	00	00
0a	00	00	0a	0a	0a	0a	0a
0b	00	0a	0b	0b	0b	0b	0b
ab	00	0a	0b	0b	ab	ab	ab
bb	00	0a	0b	ab	bb	ab	bb
c1	00	0a	0b	ab	ab	c1	c1
11	00	0a	0b	ab	bb	c1	11

The $\cdot$ table

$\rightarrow$	00	0a	0b	ab	bb	c1	11
00	11	11	11	11	11	11	11
0a	1c	11	11	11	11	11	11
0b	1c	1c	11	11	11	11	11
ab	cc	cc	c1	11	11	11	11
bb	cc	cc	c1	c1	11	c1	11
c1	00	0a	0b	bb	bb	11	11
11	00	0a	0b	ab	bb	c1	11

The $\rightarrow$ table

Table 4: Table for $(\boldsymbol{\omega_2})^{-\mathbf{1}}(\leq)^{\circ}$

Define the subalgebra

$\mathbf{r}_0 =< \{00, 0a, 0b, ab, bb, c1, 11\}; \vee,\wedge,\cdot,\rightarrow, 00, 11 >$

$\mathbf{r}_1 =< \{00, 0a, 0b, 1c, 11\}; \vee,\wedge,\cdot,\rightarrow, 00, 11 >$

$\mathbf{r}_2 =< \{00, c1, 11\}; \vee,\wedge,\cdot,\rightarrow, 00, 11 >$

$\mathbf{r}_3 =< \{00, 11\}; \vee,\wedge,\cdot,\rightarrow, 00, 11 >$

We have the following result.

Theorem 3.3. Let $\mathbf{M}_5$ be the residuated lattice given in figure 1. The topological structure $\widetilde{\mathbf{M_5}} =< M_5; G, R, \mathcal{T} >$ where $G = \{\mathrm{id}_{\mathbf{M}_5}, g\}$ and $R = \{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ yields a duality on $\mathbb{ISP}\ (\mathbf{M}_5)$.

Proof: Let $\Omega = \{\omega_1, \omega_2\}$. Then for any $x, y \in M_5$ such that $x \neq y$ either $(\omega_1 \circ \mathrm{id}_{\mathbf{M}_5})(x) \neq (\omega_1 \circ \mathrm{id}_{\mathbf{M}_5})(y)$ or $(\omega_2 \circ \mathrm{id}_{\mathbf{M}_5})(x) \neq (\omega_2 \circ \mathrm{id}_{\mathbf{M}_5})(y)$.

Hence the set

$$\{\omega \circ g: \mathbf{M} \rightarrow \mathbf{2} \mid g \in G \textit{ and } \omega \in \Omega\}$$

separates the points of M_5. Moreover by Lemma 3.2 we have $R = \{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is the set of all $\mathcal{A}$-subalgebras of $\mathbf{M}_5^2$ which are maximal in

$(\omega_1, \omega_2)^{-1}(\leq) \coloneqq \{(a, b) \in M_5^2 | \omega_1(a) \leq \omega_2(b), \omega_1, \omega_2 \in \Omega\}$.

Hence by the Piggyback Duality Theorem 2.1, $\widetilde{\mathbf{M_5}} =< M_5; G, R, \mathcal{T} >$ where $G = \{\mathrm{id}_{\mathbf{M}_5}, g\}$ and $R = \{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ yields a duality on $\mathbb{ISP}\ (\mathbf{M}_5)$.

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3.1 Entailment. Entailment is a very useful tool for reducing the size of $G \cup H \cup R$ without destroying the duality. Let $\mathcal{A}$ be the quasi-variety generated by $\mathbf{M}$ and let $\widetilde{\mathbf{M}} = < M; G, H, R, \mathcal{T} >$ be an alter ego of $\mathbf{M}$. Let s be a fixed finitary algebraic relation or (partial) operation on $\mathbf{M}$.

Given $\mathbf{A} \in \mathcal{A}$, the set $G \cup H \cup R$ is said to **entails** s on $D(\mathbf{A})$ if every continuous $(G \cup H \cup R)$-preserving map $\varphi: D(A) \to M$ also preserves s.

The set $G \cup H \cup R$ is said to **entails** s if $G \cup H \cup R$ entails s on $D(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}$.

A list of constructs for entailment is given in [1, 2.4.5]. From the list we see that the (partial) endomorphisms can entails its graph from the structure without destroying the duality. The following result shows that the relations $\mathbf{r}_2$ and $\mathbf{r}_3$ can be entailed from the structure by adding the (partial) endomorphisms of $\mathbf{M}_5$ to the structure.

Theorem. Let $\mathbf{M}_5$ be the residuated lattice given in figure 1. The topological structure $\widetilde{\mathbf{M}_5} = < M_5; g, \mathbf{r}_0, \mathbf{r}_1, h_1, h_2, \mathcal{T} >$ yields a duality on $\mathbb{ISP}\ (\mathbf{M}_5)$.

Proof: Let $h_1: M_5 \to M_5$ be a partial map defined by $h_1(0) = 0, h_1(1) = 1$. Then h_1 is a partial endomorphism on $\widetilde{\mathbf{M}_5}$. The graph of h_1, $\text{graph}(h_1) = \{00, 11\}$. Hence $\mathbf{r}_3$ is the graph of h_1. Thus we can entail $\mathbf{r}_3$ by adding $h_1 \in G$.

Again let $h_2: M_5 \to M_5$ be a partial map defined by $h_2(0) = 0, h_2(c) = h_2(1) = 1$. Then h_2 is a partial endomorphism on $\widetilde{\mathbf{M}_5}$. The graph of h_2, $\text{graph}(h_2) = \{00, c1, 11\}$. Hence $\mathbf{r}_2$ is the graph of h_2. Thus we can entail $\mathbf{r}_2$ by adding $h_2 \in G$. Hence by the construction of entailment, $\widetilde{\mathbf{M}_5}$ yields a duality on $\mathbb{ISP}\ (\mathbf{M}_5)$.

4. CONCLUSION

In this study, we successfully developed a natural duality for the quasi-variety generated by a five-element residuated lattice. Using the piggyback duality theorem as the foundational framework, we demonstrated the construction of a dual structure, establishing a clear correspondence between algebraic and topological relational representations. The unique challenges of working with a five-element lattice, such as the need for multiple carriers to ensure point separation, were addressed through tailored modifications of duality constructs.

References

1) Clark, D.M. and Davey, B.A., *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, (1998).

2) Clark, D.M. and Krauss, P.H., *Topological quasi-varieties*, Acta Sci. Math., 47 (1984), 3-39.

3) Davey, B.A. and Priestley, H.A., *Generalized Piggyback Dualities and applications to Ockham Algebras*, Houston J. Math., 13 (1987), 151-198.

4) Davey, B.A. and Priestley, H.A., *Optimal natural dualities*, Trans. Amer. Math. Soc., 338 (1993), 655-677.

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DOI: 10.5281/zenodo.14723074

5) Davey, B.A. and Werner, H., *Dualities and Equivalences for Varieties of Algebras*, Contribution to Lattice Theory (Szeged 1980) (A.P.Huhn and E.T.Schmidt,eds), Coll. Math. Soc. Ja'nos Bolyai, North Holland, Amsterdam, New York, 33 (1983), 101-275.

6) Davey, B.A. and Werner, H., *Piggyback-Dualitaten*, Bull. Austral. Math. Soc., 32 (1985), 1-32.

7) Mamun, M.A. and Talukder, M.R., *Natural Dualities for Boolean Algebras with a unary operation*, Southeast Asian Bull. Math., 32 (2008), 707-719.

8) Priestley, H.A., *Natural dualities*, Lattice theory and its applications (In Celebration of Garrett Birkhoff's 80th Birthday), Research and Exposition in Mathematics 23 (K. A. Baker and R. Wille, eds), Heldermann, Lemgo, Germany, (1995) 185-209.

9) Talukder, M. R., Mamun, M.A. and Kadir, M.S., *Natural duality for quasi-variety generated by three element residuated chain*, SUST Studies, Bangladesh, 12 no. 1 (2010), 1-6.