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A NOTE ON 1-COAXER LATTICES

C. NAG*
Department of Mathematics, Shahjalal University of Science and Technology, Sylhet, Bangladesh.
*Corresponding Author Email: chandrani-mat@sust.edu

SYED MD OMAR FARUK
Department of Mathematics, Shahjalal University of Science and Technology, Sylhet, Bangladesh.

Abstract

This paper investigates coaxer ideals within 1-distributive lattices and introduces the concept of 1-coaxer lattices. The study explores characterizations of coaxer ideals and establishes foundational properties of 1-coaxer lattices. Through an analysis of pseudocomplemented 1-distributive lattices, it provides a structured understanding of coaxer ideals, minimal and maximal ideals, and their interrelations.

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Keywords and Phrases: Lattices, 1-Distributive Lattice, Pseudocomplemented Lattice, Ideal, Filter.

1. INTRODUCTION

The notion of coaxer ideals is introduced in [1] for distributive lattices. In this paper we study coaxer ideals in 1-distributive lattices. We also refer the reader to [2, 4] for 1-distributive lattices and 0-distributive lattices.

A lattice L with 1 is called **1-distributive** if for any $a, b, c \in L$, $a \vee b = 1 = a \vee c$ implies $a \vee (b \wedge c) = 1$. The pentagonal lattice P_5 (see the diagram in Figure 1) is 1-distributive but not distributive. Thus, not every 1-distributive lattice is a distributive lattice. The diamond lattice M_3 (see Figure 1) is not 1-distributive.

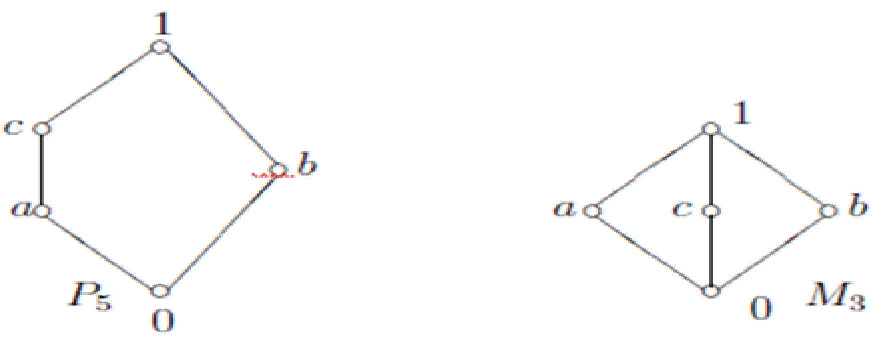

Figure 1: The pentagonal lattice and the diamond lattice

An element $a^* \in L$ is called the **psedocomplement** of $a \in L$ is the greatest element disjoint from a such that $x \leqslant a^*$ if and only if $x \wedge a = 0$.

A 1-distributive lattice L is called **pseudocomplemented 1-distributive lattice** if every element in L has a pseudocomplement. P_5 is pseudocomplemented 1-distributive lattice but M_3 is not a pseudocomplemented 1-distributive lattice(see Figure 1).

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The following well known identities (see [5, 6, 7]) are used throughout this paper.

(1) $a \leq b$ implies $b^* \leq a^*$

(2) $a \leq a^{**}$

(3) $a = a^{***}$

(4) $(a \vee b)^* = a^* \wedge b^*$

(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(6) $a \wedge (a \wedge b)^* = a \wedge b^*$

An element $a \in L$ is called **dense** if $a^* = 0$ and the set of all dense elements is denoted by $D(L)$ and is a filter of L. For every $x \in L, x \vee x^* \in D(L)$, since $(x \vee x^*)^* = 0$.

The identity (6) is used rarely (see [6] for semilattices and see [4] for lattices). An ideal I of a lattice L is called a **proper ideal** if $I \neq L$.
An ideal M of L is called a **maximal ideal** of L if M is proper and if there is a proper ideal I of L such that $M \subseteq I$ then $M = I$. A **minimal ideal** I of L is a proper ideal which is not belonging to any other proper ideal, that is, if there is a proper ideal J such that $J \subseteq I$, then $I = J$. An ideal P of L is called a **prime ideal** if for any $a, b \in L$ such that $a \wedge b \in P$ implies $a \in P$ or $b \in P$. For the background of lattices, we refer the reader to [9, 10, 11].

2. Coaxer Ideals

M. S. Rao [1] gives the definition of coaxer ideals in a pseudocomplemented distributive lattice. In this section coaxer ideals is defined in a pseudocomplemented 1-distributive lattice. Also we give the notion of 1-coaxer lattices. We discuss various properties of coaxer ideals.
Let L be a pseudocomplemented 1-distributive lattice. A non-empty subset I of L is called an **ideal** if

(i) $a \in L, b \in I$ with $a \leqslant b$ implies $a \in I$,

(ii) $a, b \in I$ implies $a \vee b \in I$.

Let L be a pseudocomplemented 1-distributive lattice, for any $a \in L$ the **coaxer** of a is the set defined as

$$(a)^\circ = \{x \in L \mid x^* \vee a = 1\}.$$

Clearly $(0)^\circ = \{0\}$ and $(1)^\circ = L$.

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Now we have an important result for this paper.

Theorem 2.1. *Let L be a pseudocomplemented 1-distributive lattice. For any $a \in L$, $(a)^\circ$ is an ideal of L.*

Proof. Since $0^* = 1$, so obviously $0 \in (a)^\circ$. Now let $x, y \in (a)^\circ$. This implies $x^* \vee a = 1$ and $y^* \vee a = 1$. Since L is 1-distributive, $a \vee (x^* \wedge y^*) = a \wedge (x \vee y)^* = 1$. So $x \vee y \in (a)^\circ$. Again let $x \in (a)^\circ$ and $r \in L$ with $r \leq x$. So $x^* \leq r^*$. This implies $1 = (x^* \vee a) \leq (r^* \vee a)$. So $(r^* \vee a) = 1$ and hence $r \in (a)^\circ$. This completes the proof.

This ideal $(a)^\circ$ for any $a \in L$ is called **coaxer ideal** of L.
Rao [1] proved that $(a)^\circ \subseteq (a]$. But in our case, if we consider the P_5 (see Figure 1), we see that $(a)^\circ = (b] \nsubseteq (a]$.

Now we have some properties of coaxer ideals.

Theorem 2.2. *Let L be a pseudocomplemented 1-distributive lattice. For any $a, b \in L$, we have*

(i) $a \leq b$ *implies that* $(a)^\circ \subseteq (b)^\circ$.
(ii) $a \vee b = 1$ *implies* $a^* \in (b)^\circ$.
(iii) $(a)^\circ \subseteq (a^{**}]$.
(iv) $(a)^\circ \cap (b)^\circ = (a \wedge b)^\circ$.
(v) $(a)^\circ = L$ *if and only if* $a = 1$.

Proof. (i) Let $a \leq b$ and let $x \in (a)^\circ$. So $x^* \vee a = 1$. Thus $1 = x^* \vee a \leq x^* \vee b$ implies $x^* \vee b = 1$ and hence $x \in (b)^\circ$.

(ii) Since $a \leq a^{**}$, we have $a \vee b = 1$ implies $a^{**} \vee b = 1$. So $a^* \in (b)^\circ$.
(iii) Let $x \in (a)^\circ$. So $x^* \vee a = 1$. This implies that $x^{**} \wedge a^* = 0$ and so $x^{**} \leq a^{**}$. Thus $x \leq a^{**}$. Hence $(a)^\circ \subseteq (a^{**}]$.

(iv) Let $x \in (a)^\circ \cap (b)^\circ$. Thus $x \in (a)^\circ$ and $x \in (b)^\circ$. This implies $x^* \vee a = 1$ and $x^* \vee b = 1$. As L is 1-distributive lattice, we have $x^* \vee (a \wedge b) = 1$. So $x \in (a \wedge b)^\circ$.

For the converse part we have, $(a \wedge b) \leq a$ and $(a \wedge b) \leq b$ and so using property (i), we found that $(a \wedge b)^\circ \leq (a)^\circ$ and $(a \wedge b)^\circ \leq (a)^\circ$. So this completes the proof.

(v) If $a = 1$, then $(a)^\circ = (1)^\circ = L$. Conversely let $(a)^\circ = L$, then $1 \in (a)^\circ$. So $1^* \vee a = 1$ and this implies $a = a \vee 0 = a \vee 1^* = 1$. This completes the proof.

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The following theorem is very important theorem which is proved for p-algebra where underlying lattice is 0-distributive lattice (see [3]). Here we give the proof of this theorem for 1-distributive lattice.

Lemma 2.3. *Let L be a pseudocomplemented 1-distributive lattice and$P$ be a prime ideal of L. Then the following conditions are equivalent:*

(i) *P is minimal;*

(ii) *$x \in P$ implies $x^* \notin P$;*

(iii) *$x \in P$ impllies $x^{**} \in P$;*

(iv) *$P \cap D(L) = \phi$.*

Proof. $(i) \implies (ii)$. Let P be a minimal prime ideal and let $x^* \in P$ for some $x \in P$. Let $D = (L \setminus P) \vee [x)$. Then D is a filter and if $0 \in D$, we get $0 = y \wedge x$ for $y \notin P$. Thus we have $y \leq x^* \in P$, which is a contradiction. So $0 \notin D$ and thus $(0] \cap D = \phi$. Now since L is 1-distributive lattice and $(0] \cap D = \phi$ by Theorem 2.3 (see [2]), there exists a prime ideal Q such that $Q \cap D = \phi$. So we have $Q \subseteq P$ and $Q \neq P$ as $x \notin Q$. So this is contradiction to the fact that P is minimal.

$(ii) \implies (iii)$. Let $x \in P$ and by (ii), $x^* \notin P$. Since $x^* \wedge x^{**} = 0$ and P is prime so $x^{**} \in P$.

$(iii) \implies (iv)$. Let $x \in P \cap D(L)$. Then $x^* = 0$ and $x^{**} \in P$ implies that $x^{**} = 1 \in P$ which is a contradiction that P is proper ideal.

$(iv) \implies (i)$. Let P is not minimal and let $Q \subset P$, where Q is a prime ideal. Let $x \in P \setminus Q$ and $x \wedge x^* = 0 \in Q$, we have $x^* \in Q \subset P$. As P is an ideal we have $x \vee x^* \in P$. As $x \vee x^* \in D(L)$ for every $x \in L$, we have $(x \vee x^*) \in P \cap D(L)$, which is a contradiction of (iv). Hence P is minimal.

Now we have the following theorem.

Theorem 2.4. *Every proper coaxer ideal of a pseudocomplemented 1- distributive lattice L is contained in a minimal prime ideal.*

Proof. Let L be a pseudocomplemented 1-distributive lattice and let $(a)^\circ$ be a proper coaxer ideal of L where $a \in L$. Let $(a)^\circ \cap D(L) \neq \phi$ and $d \in (a)^\circ \cap D(L)$. If $d \in (a)^\circ \cap D(L)$ then we have $d \in (a)^\circ$ and $d \in D(L)$. This implies $d^* \vee a = 1$ and $0 \vee a = 1$. Hence $a = 1$ and consequently $(a)^\circ = L$

Which is a contradiction to the fact that $(a)^\circ$ is a proper coaxer ideal of L. So $(a)^\circ \cap D(L) = \phi$. Then by Theorem 2.3 (see [2]), there exists a prime ideal P such that $(a)^\circ \subseteq P$ and $P \cap D(L) = \phi$. Let $x \in P$. Then $x \vee x^* \in D(L)$ and $x \vee x^* \notin P$. Since P is a ideal

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and $x \in P$, so $x^* \notin P$. Therefore P is minimal prime ideal (by Lemma 2.3) such that $(a)^\circ \subseteq P$.

Now let us discuss about a special type of ideal. If M is a maximal ideal of a pseudocomplemented 1-distributive lattice L, define

$$\pi(M) = \{x \in L \mid x^* \notin M\}.$$

Theorem 2.5. *Let L be a pseudocomplemented 1-distributive lattice and let M be a maximal ideal of L. Then $\pi(M)$ is an ideal of L such that $\pi(M) \subseteq M$.*

Proof. Let M be a maximal ideal of a pseudocomplemented 1- distributive lattice L. Then M is maximal (see [**4**]). Since $0^* = 1 \notin M$ so $0 \in \pi(M)$. So $\pi(M)$ is non-empty. Let $x, y \in \pi(M)$. Thus $x^* \notin M$ and $y^* \notin M$. So

$$x^* \wedge y^* \notin M \text{ [since } M \text{ is prime]}$$
$$\Rightarrow (x \vee y)^* \notin M$$
$$\Rightarrow (x \vee y) \in \pi(M).$$

Again let $x \in \pi(M)$ and $y \in L$ with $y \leq x$. Thus $x^* \notin M$ and consequently $(x \wedge y)^* \notin M$, otherwise $x^* \in M$. So $x \wedge y = y \in \pi(M)$. Hence $\pi(M)$ is an ideal of L.

Now let $x \in \pi(M)$. So $x^* \notin M$ and since M is a prime ideal $x \wedge x^* = 0 \in M$ implies that $x \in M$. Hence $\pi(M) \subseteq M$.

Let L be a pseudocomplemented 1-distributive lattice and let M be a maximal ideal of L. let us denote the set of all maximal ideals by μ and let $\mu_a = \{M \in \mu \mid a \in M\}$. Now we have the theorem.

Theorem 2.6. *Let L be a pseudocomplemented 1-distributive lattice and let M be a maximal ideal of L and let $a \in L$. Then $(a)^\circ = \cap_{M \in \mu^a} \pi(M)$.*

Proof. Let L be a pseudocomplemented 1-distributive lattice and let $I_0 = \cap_{M \in \mu^a} \pi(M)$. Let $x \in (a)^\circ$ and let $M \in \mu_a$. Then $x^* \vee a = 1$. Now $a \in M$ and if $x^* \in M$ then $1 \in M$, which is a contradiction. So $x^* \notin M$. So $x \in \pi(M)$ and this is true for all $M \in \mu_a$. Hence $x \in \cap_{M \in \mu^a} \pi(M)$. So $(a)^\circ \subseteq I_0$.

Conversely, let $x \in I_0$. Thus $x \in \pi(M)$ for all $M \in \mu_a$. Let $x^* \vee a \neq 1$. Then there exists a maximal ideal M_0 such that $x^* \vee a \in M_0$. Hence $x^* \in M_0$ and $a \in M_0$. This implies $x \in \pi(M_0)$. Thus $x^* \notin M_0$, which is a contradiction. Thus $x^* \vee a = 1$ and $x \in (a)^\circ$. So $I_0 \subseteq (a)^\circ$. This completes the proof.

The following result is due to [4].

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Lemma 2.7. *Let L be a lattice with 1, F be a filter of L. Then an ideal M of L disjoint from F is a maximal ideal disjoint to F if and only if for any element $a \notin M$ there exists an element $b \in M$ with $a \vee b \in F$.*

Now we have the theorem.

Lemma 2.8. *Let L be a pseudocomplemented 1-distributive lattice. Then M be a maximal ideal of L if and only if for any element $a \notin M$ there exists an element $b \in M$ with $a \vee b = 1$.*

Proof. Let L be a pseudocomplemented 1-distributive lattice and M be a maximal ideal of L. Let $a \notin M$ and $b \in M$. So $a \vee b \in [a)$ (by lemma 2.7). Since $a \notin M$, $a \vee b \notin M$ as M is an ideal. Let $a \vee b \neq 1$. So $(a \vee b]$ is a proper ideal. Then as $b \leq (a \vee b)$, $M \subseteq (a \vee b]$, which is a contradiction to the fact that M is maximal. So $a \vee b = 1$.
Conversely, Assume the condition holds and M is not maximal. Then there exists a proper ideal Q such that $M \subset Q$. Let $x \in Q \setminus M$. Then there exists $y \in M$ such that $x \vee y = 1$. Since $x \in Q$ and $y \in M \subset Q$, we have $1 = x \vee y \in Q$, which is a contradiction to the fact that Q is maximal. This completes the proof.

Now we give the definition of 1-coaxer lattices. A psedocomplemented 1-distributive lattice is called **1-coaxer lattice** if $\pi(M) = M$ for every $M \in \mu$. The pentagonal lattice $\mathcal{P}_5$ is a 1-coaxer lattice (see Figure 1).

Rao [1] proved that $(a)^\circ \vee (b)^\circ = (a \vee b)^\circ$. But in our case, if we consider $(a)^\circ$ and $(b)^\circ$ in L_1, (see Figure 2), we see that $(a)^\circ \vee (b)^\circ = (a] \vee (e] = L$ but $(a \vee b)^\circ = (c)^\circ = (a]$.

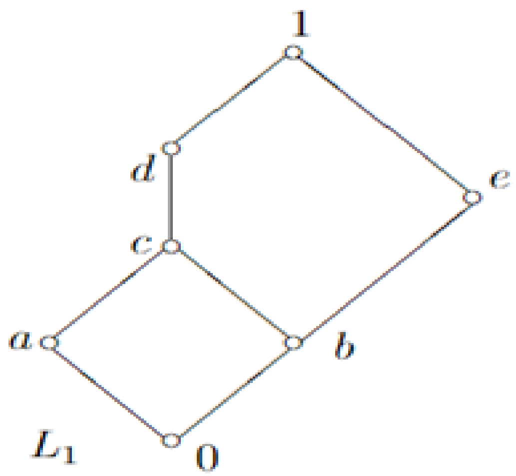

Figure 2: 1-distributive Lattice L_1

Now we have the following theorem.

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Theorem 2.9. *Let L be a pseudocomplemented 1-distributive lattice and $a, b \in L$. Then*

$$(a)^{\circ} \vee (b)^{\circ} = ((a^{*} \wedge b^{*})^{*})^{\circ}.$$

Proof. Let L be a pseudocomplemented 1-distributive lattice and $a, b \in L$. Let

$$K = ((a^{*} \wedge b^{*})^{*})^{\circ} = \{x \in L \mid x^{*} \vee (a^{*} \wedge b^{*})^{*} = 1\}.$$

Let $r \in K$ and $s \in L$ with $s \leq r$. Then $r^{*} \leq s^{*}$ implies $1 = r^{*} \vee (a^{*} \wedge b^{*})^{*} \leq s^{*} \vee (a^{*} \wedge b^{*})^{*}$. So $s \in K$. Hence K is a downset.
Now let us consider that, $r, s \in K$. So $r^{*} \vee (a^{*} \wedge b^{*})^{*} = 1$ and $s^{*} \vee (a^{*} \wedge b^{*})^{*} = 1$. As L is 1-distributive, we have $(r^{*} \wedge s^{*}) \vee (a^{*} \wedge b^{*})^{*} = 1$ and this implies that $(r \vee s)^{*} \vee (a^{*} \wedge b^{*})^{*} = 1$. So K is an ideal.
Clearly K contains $(a)^{\circ}$ and $(b)^{\circ}$, as $a \leq (a \vee b) \leq (a \vee b)^{**}$ and $b \leq (a \vee b) \leq (a \vee b)^{**}$ (by theorem 2.2). Now consider $M = (m)^{\circ}$ for any $m \in L$, be another coaxer ideal which contains $(a)^{\circ}$ and $(b)^{\circ}$. Let $r \in K$. So $r^{*} \vee (a^{*} \wedge b^{*})^{*} = 1$. Since M contains $(a)^{\circ}$ and $(b)^{\circ}$, $a \in M$ and $b \in M$. So $(a \vee b) \in M$ as M is a coaxer ideal. So $(a \vee b)^{*} \vee m = 1$ for every $m \in M$. Thus $(a \vee b)^{*} \vee m = (a \vee b)^{***} \vee m = 1$ implies that $(a \vee b)^{**} \in M$. Now as $r^{*} \vee (a^{*} \wedge b^{*})^{*} = 1$, this implies $r \in M$. So K is the smallest coaxer ideal which contains both $(a)^{\circ}$ and $(b)^{\circ}$.
This completes the proof.

Now we have the following theorem.

Theorem 2.10. *Let L be a pseudocomplemented 1-distributive lattice. Then the following are equivalent:*

(i) *L is 1-coaxer;*

(ii) *for any $a, b \in L$, $a \vee b = 1$ implies $(a)^{\circ} \vee (b)^{\circ} = L$;*

(iii) *for any $a, b \in L$, $(a)^{\circ} \vee (b)^{\circ} = ((a^{*} \wedge b^{*})^{*})^{\circ}$;*

(iv) *for any two distinct maximal ideals M and N, $\pi(M) \vee \pi(N) = L$;*

(v) *for any $M \in \mu$, M is the unique member of μ such that $\pi(M) = M$;*

(vi) *for any $M \in \mu$, $\pi(M)$ is maximal.*

Proof. $(i) \Rightarrow (ii)$. Let L be a pseudocomplemented 1-distributive lattice and L is coaxer. Let $a, b \in L$ such that $a \vee b = 1$. Suppose $(a)^{\circ} \vee (b)^{\circ} \neq L$. Then there exists a maximal ideal M such that $(a)^{\circ} \vee (b)^{\circ} \subseteq M$. This implies $(a)^{\circ} \subseteq M$ and $(b)^{\circ} \subseteq M$. So

$(a)^{\circ} \subseteq M$

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$\Rightarrow \cap_{M\in\mu^a} \pi(M) \subseteq M$ [by Theorem 2.6]

$\Rightarrow \pi(M_i) \subseteq M$ [for some $M_i \in \mu_a$ and by Theorem 2.5]

$\Rightarrow M_i \subseteq M$ [since L is coaxer]

$\Rightarrow a \in M$

$\Rightarrow a \notin L \setminus M$

Similarly $b \notin L \setminus M$. Now $L \setminus M$ is a prime filter (see [**4**]). This implies $1 = a \vee b \notin L \setminus M$. Hence $1 \in M$ and this contradicts the fact that M is proper. So $(a)^\circ \vee (b)^\circ = L$.

$(ii) \Rightarrow (iii)$. This is clear from theorem 2.9.

$(iii) \Rightarrow (iv)$. Let M and N be two distinct maximal ideals of L. Let $x \in M \setminus N$ and $y \in N \setminus M$. Then by theorem 2.8,

$$x \notin N \Rightarrow \text{ there exists } x_1 \in N \text{ such that } x \vee x_1 = 1$$

and

$$y \notin M \Rightarrow \text{ there exists } y_1 \in M \text{ such that } y \vee y_1 = 1.$$

Hence $(x \vee y_1) \vee (y \vee x_1) = (x \vee x_1) \vee (y \vee y_1) = 1$

$L = (1)^\circ = ((x \vee y_1) \vee (y \vee x_1))^\circ$

$\Rightarrow L = (((x \vee y_1) \vee (y \vee x_1))_{**})_\circ$ [since $1^{**} = 1$]

$\Rightarrow L = (x \vee y_1)^\circ \vee (y \vee x_1)^\circ$ [by (iii)]

$\Rightarrow L = (x \vee y_1)^\circ \vee (y \vee x_1)^\circ \subseteq \pi(M) \vee \pi(N)$ [since $x \vee y_1 \in M$ and $(y \vee x_1) \in N$]

$\Rightarrow L \subseteq \pi(M) \vee \pi(N)$

Thus $L = \pi(M) \vee \pi(N)$.

$(iv) \Rightarrow (v)$. Suppose condition (iv) holds. Let $M \in \mu$ and $N \in \mu$ such that $M \neq N$ and $\pi(N) \subseteq M$. But $\pi(M) \subseteq M$. So this implies that $\pi(M) \vee \pi(N) = M$ and by condition (iv), $\pi(M) \vee \pi(N) = L$, which is contradiction. So M is the unique maximal ideal such that $\pi(M) \subseteq M$.

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$(v) \Rightarrow (vi)$. Let $M \in \mu$ and let $\pi(M)$ is not maximal. Then let M_0 be another maximal ideal of L such that $\pi(M) \subseteq M_0$. But $\pi(M_0) \subseteq M_0$ implies that M is not unique maximal ideal such that $\pi(M) \subseteq M$, which is contradiction. So $\pi(M)$ is maximal.

$(vi) \Rightarrow (i)$. This is obvious from the definition of 1-coaxer lattice.

From this above theorem we have the following theorem.

Theorem 2.11. *Let L be a pseudocomplemented 1-distributive lattice. If every chain of L has at most three elements, then L is a 1-coaxer lattice.*

Proof. Let L be a pseudocomplemented 1-distributive lattice and suppose every chain of L contains atmost three elements. Let $x, y \in L$ with $x \vee y = 1$. If $x = 1$ or $y = 1$ then clearly $(x)^\circ \vee (y)^\circ = L$. Suppose $x \neq 1$ and $y \neq 1$. Then $x \wedge y \leq x < 1$. Now if $x \wedge y = x$ we have $y = (x \wedge y) \vee y = x \vee y = 1$, which is a contradiction. So $x \wedge y < x < 1$ is a three element chain. Thus $x \wedge y = 0$ and so $x \leq y^*$ and $y \leq x^*$. This implies $1 = x \vee y \leq x \vee x^*$. Thus $x \in (x)^\circ$ and similarly $y \in (y)^\circ$. Hence $1 = x \vee y \in (x)^\circ \vee (y)^\circ$. Therefore $(x)^\circ \vee (y)^\circ = L$ and L is 1-coaxer.

Rao [1] proved that the sublattice of a coaxer lattice is not always coaxer, when the lattice is distributive. So for 1-distributive lattice the sublattice of a 1-coaxer lattice is not always 1-coaxer. Now we have the following theorem.

Theorem 2.12. *Let L be a pseudocomplemented 1-distributive lattice. Then every sublattice of L is 1-coaxer if and only if for all $x, y \in L \backslash \{1\}$, $x \vee y = 1$ implies that $x \wedge y = 0$.*

Proof. At first suppose that every sublattice of L is 1-coaxer. Let $x, y \in L\backslash\{1\}$ with $x \vee y = 1$. If $x \wedge y \neq 0$ then there exists a $z \in L$ such that $0 < z < x \wedge y$. Now suppose $L_1 = \{0, z, x\wedge y, x, y, 1\}$. Then clearly L_1 is a sublattice of L. Now we have a maximal ideal $M = \{0, z, x\wedge y, x\}$ of L_1 where $\pi(M) = \{0\} \neq M$, which is a contradiction to the fact that every sublattice is 1-coaxer. So $x \wedge y = 0$.

Conversely let the condition holds and let L_1 be a sublattice of L. Let $x, y \in L_1$ with $x \vee y = 1$. Let $(a)_{L^1}{}^\circ = (a)^\circ \cap L_1$ for any $a \in L_1$. Now since $x \vee y = 1$, so $(x)_{L^1}{}^\circ \vee (y)_{L^1}{}^\circ = L_1$. Consider $x \neq 1$ and $y \neq 1$, then as per the condition $x \wedge y = 0$. Therefore $x \leq y^*$ and $y \leq x^*$. So $1 = x \vee y \leq x \vee x^*$. This implies $x \in (x)_{L^1}{}^\circ$ and similarly $y \in (y)_{L^1}{}^\circ$. Thus $1 = x \vee y \in (x)_{L^1}{}^\circ \vee (y)_{L^1}{}^\circ$. So $(x)_{L^1}{}^\circ \vee (y)_{L^1}{}^\circ = L_1$. Therefore L_1 is 1-coaxer.

3. CONCLUSION

This paper has systematically explored the concept of coaxer ideals in 1-distributive lattices and introduced the framework for 1-coaxer lattices. Through rigorous

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definitions and proofs, we established critical properties and characterizations of coaxer ideals and demonstrated their significance within the broader context of lattice theory. The results provide a deeper understanding of the structure and behavior of pseudocomplemented 1-distributive lattices, including the conditions for minimal and maximal ideals, the interplay between sublattices, and the unique attributes of 1-coaxer lattices.

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