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SOME RESULTS ON CENTERED POLYGONAL GRACEFUL GRAPHS

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Abstract

Let G be a graph with p vertices and q edges. A centered polygonal graceful labeling of a graph G is a one to one function $\lambda : V(G) \rightarrow \{0,1,2,..., C_k(q)\}$ where $C_k(q)$ is the q^{th} centered polygonal number that induces a bijection $\lambda^* : E(G) \rightarrow \{C_k(1), C_k(2),..., C_k(q)\}$ such that $\lambda^*(uv) = |\lambda(u) - \lambda(v)|$ for every edge $f^*(e) = |f(u) - f(v)|, \forall e = uv \in E(G)$. A graph which admits such a labeling is called a centered polygonal graceful graph. For $k = 3$, the above labeling gives centered triangular graceful labeling. For $k = 4$, the above labeling gives centered tetragonal graceful labeling and so on. In this paper, centered polygonal graceful labeling of some graphs are studied.

Index Terms: Banana tree, Centered polygonal graceful graph, centered polygonal graceful labeling, Centered polygonal numbers, Corona graph, F-tree, Graceful labeling, Star graph, Y-tree.

1) INTRODUCTION

We shall consider a simple, undirected and finite graph $G = (V, E)$ on $p = |V|$ vertices and $q = |E|$ edges. For all standard terminology, notations and basic definitions, we follow Harary [2] and for number theory, we follow Apostal [1]. A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges/both) then the labeling is called a vertex (edge/ total) labeling. Rosa [9] introduced β–valuation of a graph. Golomb [4] called it as graceful labeling. For a detailed survey of graph labeling, one can refer Gallian [3]. Ramesh and Syed Ali Nisaya [8] introduced some more polygonal graceful labeling of path. For more details, refer [5-8, 10]. Here, we shall recall some definitions which are used in this paper.

2) PROCEEDINGS

Definition 2.1: The star graph $K_{1,n}$ of order $n+1$ is a tree on n edges with one vertex having degree n and other vertices having degree 1.

Definition 2.2: The **Corona** $G_1 \odot G_2$ of two graphs G_1 and G_2 where G_1 has m vertices and n edges is defined as the graph G_1 obtained by taking one copy of G_1 and m copies

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of G_2, and joining by an edge the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 2.3: Y-tree on $n+1$ vertices, denoted by Y_n, is obtained from a path P_n by attaching exactly a pendant vertex to the vertices $(n-1)^{th}$ vertex of P_n.

Definition 2.4: F-tree on $n+2$ vertices, denoted by FP_n, is obtained from a path P_n by attaching exactly two pendant vertices to the vertices $n-1$ and n of P_n.

Definition 2.5: A **Banana tree** is a family of m stars $\{K_{1,n_1}, K_{1,n_2}, ..., K_{1,n_m}\},\ m \geq 1$ with a new vertex v_0 adjacent to one end vertex of each star. It is denoted by $Bt(n_1, n_2, ..., n_m)$.

Definition 2.6: A **graceful labeling** of a graph G is a one to one function $f : V(G) \to \{0,1,2,..., q\}$ that induces a bijection $f^* : E(G) \to \{1,2,..., q\}$ of the edges of G defined by $f^*(e) = |f(u) - f(v)|,\ \forall\ e = uv \in E(G)$. The graph which admits such a labeling is called a **graceful graph**.

Definition 2.7: If the n^{th} centered polygonal number is denoted by $C_k(n)$ then $C_k(n) = \frac{k}{2}[n(n-1)] + 1$, where $k \geq 3$ is the number of sides of the polygon. For $k = 3$, it gives centered triangular numbers. For $k = 4$, it gives centered tetragonal numbers and so on.

3) MAIN RESULTS

Definition 3.1: Let G be a graph with p vertices and q edges. A **centered polygonal graceful labeling** of a graph G is a one to one function $\lambda : V(G) \to \{0,1,2,..., C_k(q)\}$ where $C_k(q)$ is the q^{th} centered polygonal number that induces a bijection $\lambda^* : E(G) \to \{C_k(1), C_k(2),..., C_k(q)\}$ such that $\lambda^*(uv) = |\lambda(u) - \lambda(v)|$ for every edge $f^*(e) = |f(u) - f(v)|,\ \forall\ e = uv \in E(G)$. The graph which admits such a labeling is called a centered polygonal graceful graph. For $k = 3$, the above labeling gives centered triangular graceful labeling. For $k = 4$, the above labeling gives centered tetragonal graceful labeling and so on.

Theorem 3.2: $K_{1,n} \odot \overline{K_2}$ is a centered polygonal graceful graph for all $n \geq 1$.

Proof: Let $G = K_{1,n} \odot \overline{K_2}$ for all $n \geq 1$. Let $V(G) = \{v_i, v_{ij} : 0 \leq i \leq n,\ 1 \leq j \leq 2\}$ and $E(G) = \{v_0 v_i : 1 \leq i \leq n\} \cup \{v_i v_{ij} : 0 \leq i \leq n,\ 1 \leq j \leq 2\}$. Then G has $3n+3$ vertices and $3n+2$ edges. Let $\lambda : V(G) \to \{0,1,2,..., C_k(3n+2)\}$ be defined as follows.

$\lambda(v_0) = 0$

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$$\lambda(v_i) = \frac{3k}{2}(n-i+1)(3n-3i+2)+1,\ \ 1 \le i \le n$$
$$= C_k(3n-3i+3), \qquad 1 \le i \le n$$

$$\lambda(v_{ij}) = \lambda(v_i) - \frac{k}{2}(3n-in+n-j)(3n-in+n-j-1)-1,\ \ 1 \le i \le n,\ 1 \le j \le 2$$
$$= \lambda(v_i) - C_k(3n-in+n-j), \qquad 1 \le i \le n,\ 1 \le j \le 2$$
$$\lambda(v_{0i}) = \frac{k}{2}(3n+i)(3n+i-1)+1,\ \ i = 1,2$$
$$= C_k(3n+i), \qquad i = 1,2$$

Let $\lambda^* : E(G) \to \{C_k(1), C_k(2), ..., C_k(3n+2)\}$ be the induced edge labeling of λ. Then

$$\lambda^*(v_0 v_i) = \frac{3k}{2}(n-i+1)(3n-3i+2)+1,\ \ 1 \le i \le n$$
$$= C_k(3n-3i+3), \qquad 1 \le i \le n$$
$$\lambda^*(v_i v_{ij}) = \frac{k}{2}(3n-in+n-j)(3n-in+n-j-1)+1,\ \ 1 \le i \le n,\ 1 \le j \le 2$$
$$= C_k(3n-in+n-j), \qquad 1 \le i \le n,\ 1 \le j \le 2$$
$$\lambda^*(v_0 v_{0i}) = \frac{k}{2}(3n+i)(3n+i-1)+1,\ \ i = 1,2$$
$$= C_k(3n+i), \qquad i = 1,2$$

The induced edge labels $C_k(1), C_k(2), ..., C_k(3n+2)$ are distinct and the first $3n+2$ consecutive centered polygonal numbers. Hence $K_{1,n} \odot \overline{K_2}$ is a centered polygonal graceful graph for all $n \ge 1$.

Example 3.3: The centered pentagonal graceful labeling of $K_{1,3} \odot \overline{K_2}$ is given in figure (1).

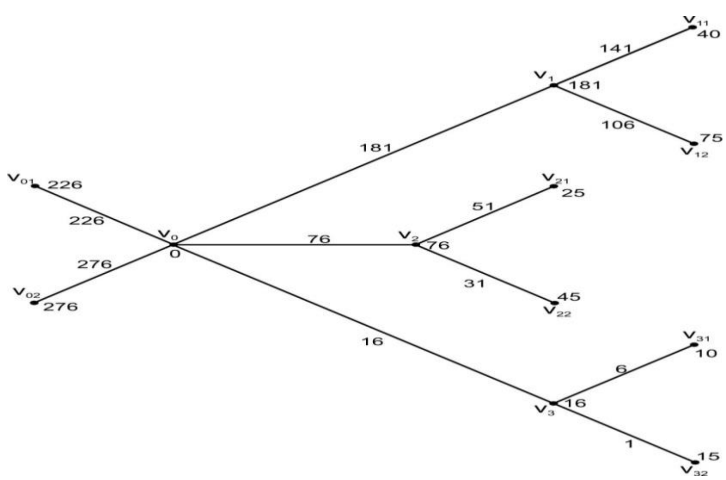

Figure (1)

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Theorem 3.4: Let G be the graph obtained by identifying the leaves of $K_{1,n}$ with the central vertex of $K_{1,2}$. Then G is a centered polygonal graceful graph for all $n \geq 1$.

Proof: Let G be the graph obtained by identifying the leaves of $K_{1,n}$ with the central vertex of $K_{1,2}$. Let $V(G) = \{v, v_i, v_{ij} : 1 \leq i \leq n \,\&\, 1 \leq j \leq 2\}$ and $E(G) = \{vv_i, v_iv_{ij} : 1 \leq i \leq n \,\&\, 1 \leq j \leq 2\}$ .Then G has $3n+1$ vertices and $3n$ edges.

Let $\lambda : V(G) \rightarrow \{0, 1, 2, ..., C_k(3n)\}$ be defined as follows.

$$\lambda(v) = 0$$

$$\lambda(v_i) = \frac{3k}{2}(n - i + 1)(3n - 3i + 2) + 1, \; 1 \leq i \leq n$$

$$= C_k(3n - 3i + 3), \; 1 \leq i \leq n$$

$$\lambda(v_{ij}) = \lambda(v_i) - \frac{k}{2}(3n - in + n - j)(3n - in + n - j - 1) - 1, \; 1 \leq i \leq n \;\; and \;\; 1 \leq j \leq 2$$

$$= \lambda(v_i) - C_k(3n - in + n - j), \; 1 \leq i \leq n \;\; and \;\; 1 \leq j \leq 2$$

Let $\lambda^* : E(G) \rightarrow \{C_k(1), C_k(2), ..., C_k(3n)\}$ be the induced edge labeling of λ. Then

$$\lambda^*(vv_i) = \frac{3k}{2}(n - i + 1)(3n - 3i + 2) + 1, \; 1 \leq i \leq n$$

$$= C_k(3n - 3i + 3), \; 1 \leq i \leq n$$

$$\lambda^*(v_iv_{ij}) = \frac{k}{2}(3n - in + n - j)(3n - in + n - j - 1) + 1, \; 1 \leq i \leq n \;\; and \;\; 1 \leq j \leq 2$$

$$= C_k(3n - in + n - j), \; 1 \leq i \leq n \;\; and \;\; 1 \leq j \leq 2$$

The induced edge labels $C_k(1), C_k(2), ..., C_k(3n)$ are distinct and the first $3n$ consecutive centered polygonal numbers. Hence G is a centered polygonal graceful graph for all $n \geq 1$.

Example 3.5: The centered hexagonal graceful labeling of the graph obtained by identifying the leaves of $K_{1,3}$ with the central vertex of $K_{1,2}$ is given in figure (2).

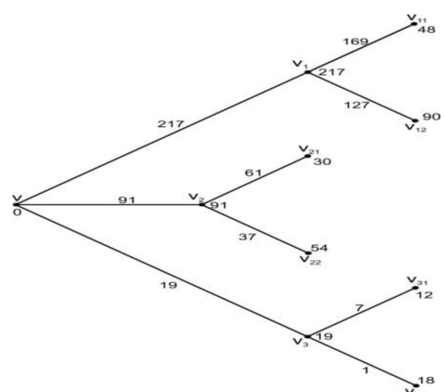

Figure (2)

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Theorem 3.6: A Y-tree Y_n for $n \geq 3$ is a centered polygonal graceful graph.

Proof: Let G be the Y-tree Y_n for $n \geq 3$. Let $V(G) = \{v, v_i : 1 \leq i \leq n\}$ and $E(G) = \{v_i v_{i+1}, vv_{n-1} : 1 \leq i \leq n-1\}$. Then G has $(n+1)$ vertices and n edges. Let $\lambda : V(G) \to \{0, 1, 2, ..., C_k(n)\}$ be defined as follows.

$\lambda(v_1) = 0$

For $2 \leq i \leq n$

$$\lambda(v_i) = \begin{cases} \lambda(v_{i-1}) - \frac{k}{2}[(n-i+2)(n-i+1)] - 1 & if \quad i \quad is \quad odd \\ \lambda(v_{i-1}) + \frac{k}{2}[(n-i+2)(n-i+1)] + 1 & if \quad i \quad is \quad even \end{cases}$$

$$= \begin{cases} \lambda(v_{i-1}) - C_k[n-i+2] & if \quad i \quad is \quad odd \\ \lambda(v_{i-1}) + C_k[n-i+2] & if \quad i \quad is \quad even \end{cases}$$

$\lambda(v) = \lambda(v_{n-1}) - 1$

Let $\lambda^* : E(G) \to \{C_k(1), C_k(2), ..., C_k(n)\}$ be the induced edge labeling of λ. Then

$$\lambda^*(v_i v_{i+1}) = \frac{k}{2}(n-i+1)(n-i) + 1, \quad 1 \leq i \leq n-1$$

$$= C_k(n-i+1), \quad 1 \leq i \leq n-1$$

$\lambda^*(vv_{n-1}) = 1$

The induced edge labels $C_k(1), C_k(2), ..., C_k(n)$ are distinct and the first n consecutive centered polygonal numbers. Therefore Y-tree is a centered polygonal graceful graph for all $n \geq 3$.

Example 3.7: The centered heptagonal graceful labeling of Y_7 is given in figure (3).

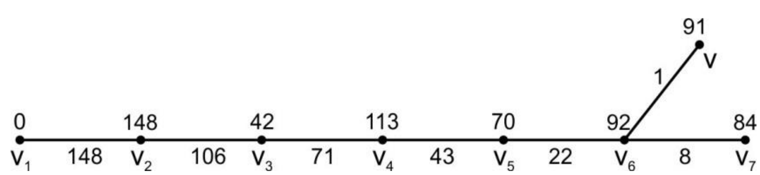

Figure (3)

Theorem 3.8: F-tree FP_n, $n \geq 3$ is a centered polygonal graceful graph.

Proof: Let $G = FP_n$, $n \geq 3$. Let $V(G) = \{u, v, v_i : 1 \leq i \leq n\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{uv_{n-1}, vv_n\}$. Then G has $n+2$ vertices and $n+1$ edges. Let $\lambda : V(G) \to \{0, 1, 2, ..., C_k(n+1)\}$ be defined as follows.

$\lambda(v_1) = 0$

For $2 \leq i \leq n$

$$\lambda(v_i) = \begin{cases} \lambda(v_{i-1}) - \frac{k}{2}[(n-i+3)(n-i+2)] - 1 & if \quad i \quad is \quad odd \\ \lambda(v_{i-1}) + \frac{k}{2}[(n-i+3)(n-i+2)] + 1 & if \quad i \quad is \quad even \end{cases}$$

$$= \begin{cases} \lambda(v_{i-1}) - C_k(n-i+3) & if \quad i \quad is \quad odd \\ \lambda(v_{i-1}) + C_k(n-i+3) & if \quad i \quad is \quad even \end{cases}$$

$\lambda(v) = \lambda(v_n) - 1$

$\lambda(u) = \lambda(v_{n-1}) - k - 1$

Let $\lambda^* : E(G) \to \{C_k(1), C_k(2), ..., C_k(n+1)\}$ be the induced edge labeling of λ. Then

$$\lambda^*(v_i v_{i+1}) = \frac{k}{2}[(n-i+2)(n-i+1)] + 1, \quad 1 \le i \le n$$
$$= C_k(n-i+2), \quad 1 \le i \le n$$

$\lambda^*(uv_{n-1}) = k + 1$

$\lambda^*(vv_n) = 1$

The induced edge labels $C_k(1), C_k(2), ..., C_k(n+1)$ are distinct and the first $n+1$ consecutive centered polygonal numbers. Hence F -tree FP_n, $n \ge 3$ is a centered polygonal graceful graph.

Example 3.9: The centered octagonal graceful labeling of FP_7 is given in figure (4).

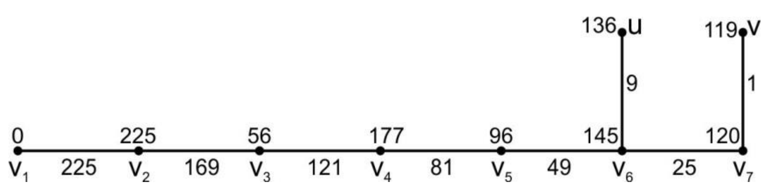

Figure (4)

Theorem 3.10: Banana tree $Bt(n_1, n_2, ..., n_m)$ is a centered polygonal graceful graph for all $n_1 = n_2 = ... = n_m \ge 3$ and $m \ge 2$.

Proof: Let $Bt(n_1, n_2, ..., n_m) = G$ for all $n_1, n_2, ..., n_m \ge 3$ and $m \ge 2$. Let $V(G) = \{v, v_i, w_i, w_{ij} : 1 \le i \le m, 1 \le j \le n_i - 1\}$ and $E(G) = \{vv_i, v_i w_i, w_i w_{ij} : 1 \le i \le m, 1 \le j \le n_i - 1\}$. Then G has $m + n_1 + n_2 + ... + n_m + 1$ vertices and $m + n_1 + n_2 + ... + n_m$ edges. Define $\lambda : V(G) \to \{0, 1, 2, ..., C_k(m + n_1 + n_2 + ... + n_m)\}$ as follows.

$\lambda(v) = 0$

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$$\lambda(v_i) = \frac{k}{2}\left[(m + n_1 + n_2 + ... + n_m - i + 1)(m + n_1 + n_2 + ... + n_m - i)\right] + 1, \quad 1 \le i \le m$$
$$= C_k(m + n_1 + n_2 + ... + n_m - i + 1), \quad 1 \le i \le m$$

$$\lambda(w_i) = \lambda(v_i) - \frac{k}{2}\left[(n_i + n_{i+1} + ... + n_m)(n_i + n_{i+1} + ... + n_m - 1)\right] - 1, \quad 1 \le i \le m$$
$$= \lambda(v_i) - C_k(n_i + n_{i+1} + ... + n_m), \quad 1 \le i \le m$$

$$\lambda(w_{ij}) = \lambda(w_i) + \frac{k}{2}\left[(n_i + n_{i+1} + ... + n_m - j)(n_i + n_{i+1} + ... + n_m - j - 1)\right] + 1, \quad 1 \le i \le m; 1 \le j \le n_i - 1$$
$$= \lambda(w_i) + C_k(n_i + n_{i+1} + ... + n_m - j), \quad 1 \le i \le m; 1 \le j \le n_i - 1$$

Clearly λ is one to one and the induced edge labels are the first $m + n_1 + n_2 + ... + n_m$ centered polygonal numbers. Hence Banana tree $Bt(n_1, n_2, ..., n_m)$ is a centered polygonal graceful graph for all $n_1 = n_2 = ... = n_m \ge 3$ and $m \ge 2$.

Example 3.11: The centered triangular graceful labeling of $Bt(5,5,5,5,5,5,5,5)$ is given in figure (5).

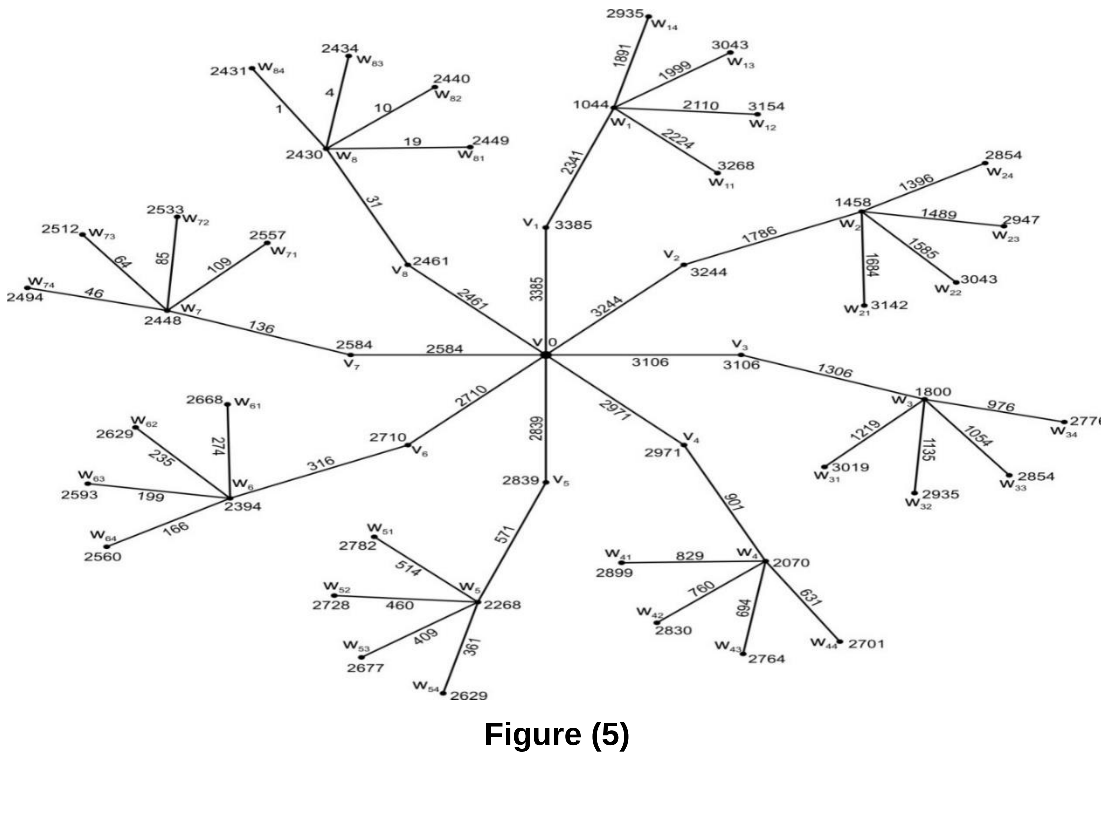

Figure (5)

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4) CONCLUSION

In this paper, we have studied about the centered polygonal graceful labeling of some tree related graphs. The results proved in this paper are novel. Examples are provided in each theorem for better understanding of the labeling pattern in each theorem.

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