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PAIRWISE D_α –OPEN AND D_α –CLOSED FUNCTIONS IN BITOPOLOGICAL SPACES

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Abstract

D_α –Open Sets That Are Introduced And Analyzed In Topological Spaces By Sayed O.R And Khalil A.M [1] Are Precisely Between The Class Of All A –Open Sets And G –Open Sets. In This Paper, We Introduce The Concepts Of D_A –Open And D_A –Closed Functions In Bitopological Spaces. We Studied When The Graph $G(F)$ Is P –Strongly Closed And $P-D_A$ –Closed Subsets Of The Bitopological Space (X, T_1, T_2). In Addition, We Define D_A^I –Interior Of A Subset A Of X And D_A^I –Closure And Prove Some Theorems Concerning These Concepts.

Index Terms: $P-D_A$ –Continuous Function, $P-D_A$ –Graph, $P-D_A$ –Open Set.

1) INTRODUCTION

Generalized open sets are a key component of general topology, and many mathematicians worldwide are now concentrating their study on them. The study of differently modified versions of continuity, separation axioms and other concepts using extended open sets is a major issue in real analysis and general topology. Typically, the notions of α –open sets, initiated by Njstad [1] in 1965, and the generalized closed or (g –closed) subsets, presented by Levine [2] in 1970, are the most well-known and also represent sources of inspiration and have been extensively researched in the literature since then.

Since then, lots of mathematicians have focused on using α –open sets and generalized closed sets to generalize different ideas in topology. Kelly [3] pioneered the study of bitopological spaces in a London mathematics society paper titled bitopological spaces in 1963, and since then, several papers have been offered to expand topological notions to bitopological ones. The triple(X, τ_1, τ_2), or simply X, is said to be a bitopological space

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if X is a non-empty set endowed with two topologies τ_1 and τ_2. Cs'asz'ar [4] proposed the notion of generalized topological spaces in the twentieth century, which has been researched by lots of mathematicians all over the world. As a result, mathematicians adopted a new approach, seeking to extend numerous topological concepts to this new domain.

Dunham [5] in 1982 used g-closed subsets of X to define a new closure operator and hence a new topological space (X, τ^*) where he transferred regularity conditions of a topological space (X, τ) to separation conditions of the new topological space (X, τ^*). Kasahara [8] introduced α −closed graphs of an operation and initiated the idea of an operation on topological spaces. Ogata [9] introduced the operation α as γ −operation and presented the notion of τ_γ which is the set of all γ-open sets.

2) PAIRWISE D_α − SETS

Definition 2.1: A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1 - g$ −closed if

$C_1(A) \subset O_2(A) \subset \tau_2$ And A is $\tau_2 - g$ −closed if $C_2(A) \subset O_1(A) \subset \tau_1$. If A is $\tau_1 - g$ −closed and $\tau_2 - g$ −closed, then it is called a pairwise g −closed set and denoted by $p - g$ −closed. $X - A$ is p −open.

Theorem 2.2: [1] g −closedness property is closed under arbitrary union provided that every g −closed set is closed.

Definition 2.3: In a bitopological space (X, τ_1, τ_2), if A is a subset of X then $\forall i, j = 1,2, i \neq j$:

i. If $A \subset int_i (cl_j (int_i (A)))$, then A is called $\tau_i - \alpha$ −open.
ii. If $A \subset cl_i (int_j(cl_i (A)))$, then A is called $\tau_i - \alpha$ −closed.
iii. A is τ_i −generalized closed ($\tau_i - g$ −closed) if $cl_i (A) \subset U$ for some τ_i −open subset U of (X, τ_i).
iv. $GO(X) = \{A: A \text{ is } \tau_i - g - \text{open}\}$
v. $GC(X) = \{A: A \text{ is } \tau_i - g - \text{closed}\}$
vi. If $O = \cap \{O: O \text{ is } \tau_i - \alpha - \text{open}, O \subset A\}$, then O is denoted by $int_\alpha(A)$.
vii. If $F = \cap \{F: F \text{ is } \tau_i - \alpha - \text{closed}, A \subset F\}$, then F is denoted by $cl_\alpha(A)$.
viii. $cl_i^*(A)$ denotes $\cap \{M: M \text{ is } \tau_i - g - \text{closed}, A \subset M\}$.
ix. $\tau_i - \alpha O(X) = \{U: U \text{ is } i - \alpha - \text{open in } X\}$.
x. $\tau_i - \alpha C(X) = \{F: F \text{ is } i - \alpha - \text{open in } X\}$.
xi. $\tau_i - \alpha O(X, x) = \{U: x \in U \in \tau_i - \alpha O(X)\}$.
xii. $i - \alpha C(X, x) = \{F: x \in F \in \tau_i - \alpha C(X)\}$.

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Definition 2.4: If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, then:

i. The subspace $\{(x, f(x)): x \in X\}$ of $X \times Y$ is the graph of f [6] and denoted by $G(f)$.
ii. f is p −closed if $G(f)$ is a (τ_i, σ_i) −closed subset of $X \times Y$ $\forall i = 1,2$.
iii. f is a p-strongly closed graph if $\forall\ (x, y) \in G(f), \exists U$ an τ_i −open subset of X and V a τ_j −open subset of Y such that $x \in V$ and $(U \times cl_j(V)) \cap G(f) = \phi\ \forall i, j = 1,2, i \neq j$.
iv. f is p −strongly α −closed graph if $\forall (x, y) \in X \times Y - G(f)$, $\exists U$ an $\tau_i - \alpha$ −open subset of X and V a $\tau_j - \alpha$ −open subset of Y such that $(U \times cl_j(V)) \cap G(f) = \phi\ \forall i, j = 1,2, i \neq j$.

Lemma 2.5: The graph $G(f)$ is p −strongly closed iff $\forall (x, y) \notin G(f)$, $\exists U$ a $\tau_i - \alpha$ −open subset of X and W a $\tau_j - \alpha$ −open subset of Y such that containing x and y (respectively) such that $f(U) \cap cl_j(W) = \phi\ \forall i, j = 1,2, i \neq j$.

Definition 2.6: A subset A of (X, τ_1, τ_2) is $\tau_i - D_\alpha$ −closed if $cl_i^*\left(int_j\big(cl_i(A)\big)\right)$ $\forall i, j = 1,2, i \neq j$. If A is $\tau_i - D_\alpha$ −closed, then $X - A$ is $\tau_i - D_\alpha$ −open .

If A is $\tau_i - D_\alpha$ −closed $\forall i = 1,2$, then A is $p - D_\alpha$ −closed. The set of all $p - D_\alpha$ −closed subsets of (X, τ_1, τ_2) is denoted by $p - D_\alpha - C(X)$.

Theorem 2.7: (i) A subset F of (X, τ_1, τ_2) is an $\tau_i - D_\alpha$ −closed if it is $\tau_i - \alpha$ −closed $\forall i = 1,2$.

(ii) A subset F of (X, τ_1, τ_2) is $\tau_i - D_\alpha$ −closed if it is $\tau_i - g$ −closed $\forall i = 1,2$.

Proof: (i) Let F be a $\tau_i - \alpha$ −closed subset of (X, τ_1, τ_2) $\forall i = 1,2$.

Then $cl_i^*\ (F) \subset cl_i(F)$ [7].

Hence, $cl_i(int_j\big(cl_i(F)\big)) \subset F$.

So, $cl_i^*\left(int_j\big(cl_i(F)\big)\right) \subset cl_i^*\left(int_j\big(cl_i^*(F)\big)\right) \subset F\ \forall i, j = 1,2, i \neq j$.

Therefore, F is $\tau_i - D_\alpha$ −closed

(ii) Let F be a $\tau_i - g$ −closed subset of X $\forall i = 1,2$.

Then, $cl_i^*(F) = F$. [7]

So, $int_i\left(cl_j^*(F)\right) \subset cl_j^*(F)\ \forall i, j = 1,2, i \neq j$.

Hence, $cl_i^*\left(int_j\big(cl_i^*(F)\big)\right) \subset cl_i^*\big(cl_i^*(F)\big) \subset cl_i^*(F) = F$.

Thus, F is $\tau_i - D_\alpha$ −closed.

Lemma 2.8: If F is $\tau_i - g$ −closed subset of (X, τ_1, τ_2) such that $int_j\big(cl_i^*(A)\big) \subset A \subset F$ for some $A \subset X$, then A is $\tau_i - D_\alpha$ −closed $\forall i, j = 1,2, i \neq j$.

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Proof: $cl_i^*(F) = F$ since F is $\tau_i - g$ −closed subset of X $\forall i = 1,2$.

Hence, $cl_i^*\left(int_j\left(cl_i^*(A)\right)\right) \subset cl_i^*\left(int_i^*(A)\right) \subset A\ \forall i,j = 1,2\ , i \neq j$.

Thus, A is $\tau_i - D_\alpha$ −closed.

Theorem 2.9: In the bitopological space (X, τ_1, τ_2), arbitrary intersection of $\tau_i - D_\alpha$ −closed sets is $\tau_i - D_\alpha$ −closed.

Proof: If $F = \{F_\beta : \beta \in \Gamma\}$ is a family of $\tau_i - D_\alpha$ −closed subsets of (X, τ_1, τ_2).

Then, $cl_i^*\left(int_j\left(cl_i^*(F_\beta)\right)\right) \subset F_\beta\ \forall \beta \in \Gamma$.

But $\bigcap_{\beta \in \Gamma} F_\beta \subset \beta \in \Gamma\ \forall \beta \in \Gamma$.

So, $cl_i^*\left(int_j\left(cl_i^*(F_\beta)\right)\right) \subset \bigcap_{\beta \in \Gamma} cl_i F_\beta$

Hence, $cl_i^*\left(int_j\left(cl_i^*(F_\beta)\right)\right) \subset \bigcap_{\beta \in \Gamma} cl_i F_\beta$

$$\subset cl_i^*\left(int_j\left(cl_i^*(F_\beta)\right)\right)$$

$$\subset \bigcap_{\beta \in \Gamma} cl_i F_\beta \quad \forall \beta \in \Gamma.$$

Thus, $\bigcap_{\beta \in \Gamma} F_\beta$ is $\tau_i - D_\alpha$ −closed.

Corollary 2.10: (i) If F_1 and F_2 are two subsets of (X, τ_1, τ_2) such that F_1 is $\tau_i - D_\alpha$ −closed and F_2 is $\tau_i - \alpha$ −closed, then $F_1 \cap F_2$ is $\tau_i - D_\alpha$ −closed $\forall i = 1,2$.

(ii) If F_1 and F_2 are two subsets of (X, τ_1, τ_2) such that F_1 is $\tau_i - D_\alpha$ −closed and F_2 is $\tau_i - g$ −closed, then $F_1 \cap F_2$ is $\tau_i - D_\alpha$ −closed $\forall i = 1,2$.

Remark 2.11: In the bitopological space (X, τ_1, τ_2), if A is a subset of X, then $\forall i = 1,2$:

(i) $X - int_i(X - A) = cl_i(A)$.

(ii) $X - cl_i(X - A) = int_i(A)$.

Theorem 2.12: In (X, τ_1, τ_2), a subset A is $\tau_i - D_\alpha$ −open iff

$A \subset int_i(cl_j^*\left(int_i(A)\right))\ \forall i,j = 1,2\ , i \neq j$.

Proof: For the necessary part, if A is an $\tau_i - D_\alpha$ −open subset of X $\forall i = 1,2$.

Then, $X - A$ is $\tau_i - D_\alpha$ −closed and $cl_i^*(int_i\left(cl_i^*(A)\right) \subset X - A$.

So, $A \subset int_i^*\left(cl_j\left(int_i^*(A)\right)\right)\ \forall i,j = 1,2\ , i \neq j$.

For the sufficient part, if $\subset int_i^*\left(cl_j\left(int_i^*(A)\right)\right)\ \forall i,j = 1,2\ , i \neq j$.

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Then, , $X - int_i^*\left(cl_j\left(int_i^*(A)\right)\right) \subset X - A$.

So, $int_i^*\left(cl_j\left(int_i^*(A)\right)\right) \subset X - A$

Hence, $X - A$ is $\tau_i - D_\alpha$ −closed.

Thus, A is $\tau_i - D_\alpha$ −open.

Corollary 2.13: For (X, τ_1, τ_2), a subset A is $\tau_i - D_\alpha$ −open if there exists U a $\tau_i - g$ −open subset such that $U \subset A \subset int_i^*\left(cl_j(U)\right)$, then A is $\tau_i - D_\alpha$ −open $\forall i, j = 1,2\,, i \neq j$.

Proof: Suppose that U is an $\tau_i - g$ −open subset of X $\forall i = 1,2$.

Then, $X - U$ is $\tau_i - g$ −closed and $X - int_i^*\left(cl_j(X - U)\right) \subset X - A \subset X - U$

Thus, $cl_i^*\left(int_j(X - U)\right) \subset X - A \subset X - U$.

So, $X - A$ is $\tau_i - D_\alpha$ −closed.

Hence, A is $\tau_i - D_\alpha$ −open $\forall i = 1,2$.

Theorem 2.14: For (X, τ_1, τ_2), every $\tau_i - \alpha$ −open ($\tau_i - g$ −open) is $\tau_i - D_\alpha$ −open $\forall i =$ 1,2.

Corollary 2.15: For the space (X, τ_1, τ_2):

(i) Arbitrary union of $\tau_i - D_\alpha$ −open set is $\tau_i - D_\alpha$ −open $\forall i = 1,2$.

(ii) If A is an $\tau_i - D_\alpha$ −open subset of X and B is an $\tau_i - \alpha$ −open set, then $A \cup B$ is $\tau_i - D_\alpha$ −open.

(iii) If A is an $\tau_i - D_\alpha$ −open subset of X and V is an $\tau_i - g$ −open set, then $A \cup V$ is $\tau_i - D_\alpha$ −open.

Definition 2.16: In the bitopological space (X, τ_1, τ_2), if $A \subset X$, then $\forall i = 1,2$:

i. The D_α^i −interior of a subset A of X, denoted by $D_\alpha^i - int_i(A)$, equals $\bigcup_{\gamma \in \Gamma}\{U_\gamma : U_\gamma \in \tau_i - D_\alpha O(X),\ U_\gamma \subset A\}$.

ii. The D_α^i −closure of a subset A of X, denoted by $D_\alpha^i - cl_i(A)$ equals $\bigcap_{\gamma \in \Gamma}\{F_\gamma :\ F_\gamma \in \tau_i - D_\alpha C(X), A \subset F_\gamma\}$.

Lemma 2.17: In the bitopological space (X, τ_1, τ_2), if $A \subset X$, then $\forall i = 1,2$:

(i) $X - (D_\alpha^i - int_i(A)) = D_\alpha^i - cl_i(A)$

(ii) $X - (D_\alpha^i - cl_i(A)) = D_\alpha^i - int_i(A)$

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Theorem 2.18: For the bitopological space (X, τ_1, τ_2), if $A, B \subset X$, then $\forall i, j = 1,2\,, i \neq j$, then:

(i) $D_\alpha^i - int_i(\phi) = \phi$ and $D_\alpha^i - int_i(X) = X$.

(ii) A is $p - D_\alpha$ open iff $D_\alpha^i - int_j(A) = A$ and $D_\alpha^j - int_i(A) = A$ $\forall i, j = 1,2 \quad i \neq j$.

(iii) $\tau_i - \alpha - int_j(A) \subset D_\alpha^i - int_j(A) \subset A$.

(iv) $int_i^*(A) \subset D_\alpha^i - int_j(A)$

(v) $D_\alpha^i - int_i\left(D_\alpha^i - int_j(A)\right)$=$D_\alpha^i - int_j(A)$.

(vi) If $A \subset B$, then $D_\alpha^i - int_j(A) \subset D_\alpha^i - int_j(B)$

(vii) $(D_\alpha^i - int_j(A))$∪$(D_\alpha^i - int_j(B))$⊂ $D_\alpha^i - int_j(A \cup B)$

(viii) $D_\alpha^i - int_j\ (A \cap B)$ ⊂$(D_\alpha^i - int_j(A))$∩$(D_\alpha^i - int_j(B))$.

Theorem 2.19: In the bitopological space (X, τ_1, τ_2), if $A \subset X$, then $\forall i, j = 1,2\,, i \neq j$:

(i) $D_\alpha^i - int_j(A) = A \cap int_i^*\left(cl_j\big(int_i^*(A)\big)\right)$.

(ii) $D_\alpha^i - cl_j(A) = A \cup int_i^*\left(cl_j\big(int_i^*(A)\big)\right)$.

Proof: (i) $D_\alpha^i - int_j(A)$ is $p - D_\alpha -$ open and $D_\alpha^i - int_j(A) \subset A$.

Hence, $D_\alpha^i - int_j(A) \subset int_i^*\left(cl_j\left(D_\alpha^i - int_j(A)\right)\right)$

$$\subset int_i^*\left(cl_j(int_i^*(A)\right).$$

Now, , $D_\alpha^i - int_j(A) \subset A \cup int_i^*\left(cl_j\big(int_i^*(A)\big)\right)$

So, $A \cup int_i^*\left(cl_j\big(int_i^*(A)\big)\right)$ is $p - D_\alpha -$ open

$A \cup int_i^*\left(cl_j\big(int_i^*(A)\big)\right) \subset D_\alpha^i - int_j(A)$.

Thus, $D_\alpha^i - int_j(A) = A \cap int_i^*\left(cl_j\big(int_i^*(A)\big)\right)$.

(ii) $D_\alpha^i - cl_j(A) = X - int_j(X - A) =$

$$= X - (X - A)$$

$$= X - (X - A) \cup (X - int_i^*\left(cl_j\big(int_i^*(X - A)\big)\right))$$

$$= A \cup cl_i^*\left(int_j\big(cl_i^*(A)\big)\right)$$

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3) PAIRWISE D_α −CONTINUOUS FUNCTIONS

Definition 3.1: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, then the function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $\tau_i - D_\alpha$ −continuous if the inverse image of each σ_i −open set in Y is $\tau_i - D_\alpha$ −open in X $\forall i, j = 1,2, i \neq j$.

Lemma 3.2: (i) Every $\tau_i - \alpha$ −continuous function is $\tau_j - D_\alpha$ −continuous $\forall i, j = 1,2, i \neq j$.

(ii) Every $\tau_i - g$ −continuous function is $\tau_i - D_\alpha$ −continuous $\forall i, j = 1,2, i \neq j$ [10].

Theorem 3.3: In (X, τ_1, τ_2) and (Y, σ_1, σ_2), if $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, then $\forall i, j = 1,2, i \neq j$ and $\forall A \subset X, B \subset Y$, the following are equivalent:

(i) f is $\tau_i - D_\alpha$ −continuous.

(ii) $\forall x \in X$ and for each τ_j −open subset V of Y such that $f(x) \in V$, there exists a $\tau_j - D_\alpha$ −open subset U of X containing such that $x \in X$ and $f(U) \subset V$.

(iii) Inverse image of each σ_1 −closed subset of Y is τ_2 −closed subset of X.

(iv) $f(D_\alpha^i - int_j(A)) \subset cl_j(f(A))$

(v) $D_\alpha^i - cl_j(\, f^{-1}(B)) \subset f^{-1}(cl_j(B))$.

(vi) $f^{-1}(int_i(B)) \subset D_\alpha^i - int_j(f^{-1}(B))$.

Proof:

(i)→(ii) $f^{-1}(V) \in p - D_\alpha O(X)$ $\forall p$ −open subset of Y.

Now, if $x \in f^{-1}(V)$, then $f(f^{-1}(V)) \subset V$

(ii)→(iii) Direct

(iii)→(iv) If F is a p −closed subset of Y such that $F \subset f(A)$.

Then $A \subset f^{-1}(F)$ is $p - D_\alpha -$ closed subset of X.

Hence, $D_\alpha^i - cl_j(A) \subset D_\alpha^i - cl_j\ (f^{-1}(F)) = f^{-1}(F)$.

Thus, $f(D_\alpha^i - cl_j(A)) \subset F$.

Therefore, $f(D_\alpha^i - cl_j(f(A)) \subset cl_j(f(A)$.

(iv)→(v) Let B be a subset of Y.

Then $f(D_\alpha^i - cl_j(f^{-1}(B)) \subset cl_i(f(f^{-1}(B))$

$\subset cl_i(B)$.

So, $D_\alpha^i - cl_j((f^{-1}(B)) \subset cl_j f^{-1}((B)$.

Hence, $D_\alpha^i - cl_j((f^{-1}(B\ )) \subset f^{-1}(cl_j(B))$.

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(v)→(vi) $D_\alpha^i - cl_j\ (f^{-1}(Y-B)) \subset f^{-1}(D_\alpha^i - cl_j\ (Y-B))$.

Then $D_\alpha^i - cl_j\ (X - f^{-1}(B)) \subset f^{-1}(Y - int_j(B))$.

So, $-D_\alpha^i - int_j\ (f^{-1}(B)) \subset X - f^{-1}(int_j(B))$.

Thus, $f^{-1}(int_j(B)) \subset D_\alpha^i - int_j(f^{-1}(B))$

(vi)→(i) Let V be a τ_i −open subset of Y [11].

Then $f^{-1}\big(int_i(V)\big) \subset D_\alpha^i - int_j\big(f^{-1}(V)\big)$.

So, $f^{-1}(int_j(V)) \subset D_\alpha^i - int_j(f^{-1}(V))$.

Thus, $f^{-1}(V)$ is an $\tau_i - D_\alpha$ −open.

Consequently, f is $p - D_\alpha$ −continuous.

Theorem 3.4: The composition of $\tau_i - D_\alpha$ −continuous function and τ_i −continuous function in the bitopological space (X, τ_1, τ_2) is $\tau_i - D_\alpha$ −continuous $\forall i = 1,2$.

Definition 3.5: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, then the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has D_α −closed graph if $\forall (x, y) \in (X \times Y) - G(f)$,

$\exists U \in \tau_i - D_\alpha O(X, x)$ and $V \in GO(Y, y)$: $(U \times cl_i^*(V)) \cap G(f) = \phi\ \forall i = 1,2$.

Lemma 3.6: In (X, τ_1, τ_2), every closed graph is $\tau_i - D_\alpha$ −closed $\forall i = 1,2$.

Theorem 3.7: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, then the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\tau_i - D_\alpha$ −closed graph iff $\forall (x, y) \in (X \times Y) - G(f)$,

$\exists U \in \tau_i - D_\alpha O(X, x)$ and $V \in GO(Y, y)$: $(U \times cl_i^*(V)) \cap G(f) = \phi\ \forall i = 1,2$.

Proof: For the necessary part, if $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is D_α −closed graph.

Then $\forall (x, y) \in (X \times Y) - G(f)$, $\exists U \in \tau_i - D_\alpha O(X, x)$ and $V \in GO(Y, y)$:

$(U \times cl_i^*(V)) \cap G(f) = \phi$.

So, $f_i(x) \in f_i(U)$ and $y \in cl_i^*(V)$.

But $y \neq f_i(x)$, hence $f_i(U) \cap cl_i^*(V) = \phi$.

For the sufficient part, suppose that $(x, y) \in (X \times Y) - G(f)$

$\exists U \in \tau_i - D_\alpha O(X, x)$ and $V \in GO(Y, y)$: $(U \times cl_i^*(V)) \cap G(f) = \phi\ \forall i = 1,2$.

Then $f_i(x) \neq y$ and $f_i(U) \cap cl_i^*(V) = \phi$.

Theorem 3.8: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, then the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\tau_i - D_\alpha$ −closed graph if $\forall (x, y) \in (X \times Y) - G(f)$,

$\exists U \in \tau_i - D_\alpha O(X, x)$ and $V \in \tau_j - D_\alpha(Y, y)$:(U×$cl_j^*(V) \cap G(f) = \phi$.

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Proof: If f is an D_α −closed graph, then for $(x,y) \in (X \times Y) - G(f)$, $\exists U \in \tau_i - D_\alpha O(X,x)$ and $V \in GO(Y,y)$ [13].

But $\tau_i - g$ −subset is $\tau_i - D_\alpha$ −open, then $D_\alpha^i - cl_j(V) \subset cl_j(V)$.

Hence, $(U \times D_\alpha^i - cl_j(V)) \cap G(f) = \phi \ \forall i = 1,2$ [14].

Theorem 3.9: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, then the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is D_α −closed graph $\forall (x,y) \in (X \times Y) - G(f), \exists U \in D_\alpha O(X,x)$ and

V∈D_α (Y,y)☹$U \times D_\alpha^i - cl_j(V)) \cap G(f) = \phi$ if $\forall (x,y) \in (X \times Y) - G(f)$,

$\exists U \in \tau_i - D_\alpha O(X,x)$ and $V \in \tau_i - D_\alpha - (Y,y)$: $f_i(U) \cap D_\alpha^i - cl_j(V) = \phi$.

Definition 3.10: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, then the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is has a strongly D_α −closed graph if $\forall (x,y) \in (X \times Y) - G(f)$,

$\exists U \in D_\alpha O(X,x)$ and $V \in O(Y,y)$: $(U \times cl_i(V)) \cap G(f) = \phi \ \forall i = 1,2$.

Remark 3.11: (i) A strongly D_α −closed graph is D_α −closed in (X, τ_1, τ_2).

(ii) Every strongly $\tau_i - \alpha$ −closed graph is strongly D_α −closed graph in (X, τ_1, τ_2) $\forall i = 1,2$.

Theorem 3.12: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces and fot the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following are equivalent:

(i) f has strongly D_α −closed graph.

(ii) $\forall (x,y) \in (X \times Y) - G(f)$, $\exists U \in D_\alpha O(X,x)$ and $V \in O(Y,y)$: $f(U) \cap cl_i(V) = \phi \ \forall i = 1,2$.

(iii) $\forall (x,y) \in (X \times Y) - G(f), \exists U \in D_\alpha O(X,x)$ and $V \in O(Y,y)$: $(U \times cl_j(V)) \cap G(f) = \phi$.

Corollary 3.12: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces and the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is strongly D_α −closed graph, then $\forall x \in X$,

$f(x) = \cap \{cl_i(W): W \in p - D_\alpha O(X,x)\}$.

Proof: Suppose that if $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is strongly D_α −closed graph, then $\exists y \neq f(x): y \in \cap \{cl_i(W): W \in p - D_\alpha O(X,x)\}$.

So, $y \in cl_i(f(W))$ for some $W \in p - D_\alpha O(X,x)$.

Hence, $\forall V \in \alpha O(Y,y)$, $V \cap f(W) \neq \phi$.

Thus, $f(W) \neq \phi$ and $f(W) \subset V \subset f(V) \cap cl_\alpha^i(V)$ which contradicts that f is strongly D_α −closed graph.

As a consequence, $x \in X$, $f(x) = \cap \{cl_i(W): W \in p - D_\alpha O(X,x)\}$.

Theorem 3.13: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces, the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $p - D_\alpha$ −continuous, and Y is p −Housdorff, then $G(f)$ is strongly $p - D_\alpha$ −closed.

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Proof: Suppose that $(x, y) \in (X \times Y) - G(f)$. Since Y is p −Housdorff,

$\exists W \in O(Y, y)$: $f(x) \notin cl_i(W)\ \forall i = 1,2$.

But $cl_i(W)$ is τ_i −closed, so $Y - cl_i(W) \in O(Y, y)$ and $\exists U \in D_\alpha O(X, x)$: $f(U) \subset Y - cl_i(W)$.

Hence, $f(U) \cap cl_i(W) = \phi$.

Thus, $G(f)$ is strongly D_α −closed [12].

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